Financial Prelim 2016 Solutions

Questions by Anmol Bhandari and Jan Werner
Solutions by Isaac Swift

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Section I

Exercise (I.1)**.** For a non-negative random variable m such that $\mathbb{E}[m] = 1$ show that $\mathbb{E}[m \log(m] \geq 0$.

Solution. First define $\phi(m) = m \log(m)$. Note that $\phi''(m) = \frac{1}{m} > 0$ since m is non-negative. So, ϕ is convex in m. Since ϕ is everywhere convex, it always lies above its first-order Taylor expansion.

$$\begin{aligned} \phi(m) &\geq \phi(1) + \phi'(1)(m-1) \\ &= 1 * \log(1) + (\log(1) + 1)(m-1) \\ &= m - 1 \end{aligned}$$

We can now take the expectation of both sides of the equation to get the result. $\mathbb{E}[m \log(m] = \mathbb{E}[\phi(m] \geq \mathbb{E}[m-1] = 0$. ☐

Exercise (I.2)**.** Consider security markets extending over T periods where $T < \infty$, with J long-lived securities that may pay dividends at any subset of dates from 1 to T. There are no constraints on portfolio holdings. Assuming that security prices are arbitrage free and there is a one-period risk-free return at every date, show that discounted gains on every security have the martingale property under risk-neutral probabilities.

Solution. This is an immediate implication from the definitions in Jan Werner's notes. First the discount factor.

$$\rho(s_t) = \prod_{\tau=1}^{t} \frac{1}{\bar{r}(s_\tau)}$$

Take the event prices to be $q(s_t)$. Define the risk neutral probabilities by

$$\pi^*(s_t) = \frac{q(s_t)}{\rho(s_t)}.$$

From the definition of event prices, we have

$$q(s_t)p_j(s_t) = \sum_{s_{t+1}\subset s_t} q(s_{t+1})p_j(s_{t+1}) + q(s_{t+1})x_j(s_{t+1}).$$

This implies

$$\begin{aligned} p_j(s_t) &= \sum_{s_{t+1}\subset s_t} \frac{1}{\bar{r}(s_{t+1})}\frac{q(s_{t+1})}{\rho(s_{t+1})}\frac{\rho(s_t)}{q(s_t)}(p_j(s_t)+x_j(s_t)) \\ &= \frac{1}{\bar{r}(s_{t+1})}\mathbb{E}_{\pi^*(s_{t+1}|s_t)}[p_j(s_{t+1})+x_j(s_{t+1})]. \end{aligned}$$

Finally, take the definition of discounted gain.

$$d_j(s_t) = \rho(s_t)p_j(s_t) + \sum_{\tau=1}^{t}\rho(s_\tau)x_j(s_\tau)$$

Now we've finished definitions and can solve the actual problem. I'm going to use shorthand notation from this point on.

Take any two dates $T > t$.

$$\begin{aligned} \mathbb{E}_t^*[d_{j,T}] &= \mathbb{E}_t^*\left[\rho_T p_{j,T} + \sum_{\tau=1}^{T}\rho_\tau x_{j,\tau}\right] \\ &= \mathbb{E}_t^*\left[\rho_T(p_{j,T}+x_{j,T})\right] + \mathbb{E}_t^*\left[\sum_{\tau=t+1}^{T-1}\rho_\tau x_{j,\tau}\right] + \sum_{\tau=1}^{t}\rho_\tau x_{j,\tau} \end{aligned}$$

By the Law of Iterated Expectations,

$$\begin{aligned} &= \mathbb{E}_t^*\left[\rho_{T-1}\frac{1}{\bar{r}_T}\mathbb{E}_{T-1}^*[p_{j,T}+x_{j,T}]\right] + \mathbb{E}_t^*\left[\sum_{\tau=t+1}^{T-1}\rho_\tau x_{j,\tau}\right] + \sum_{\tau=1}^{t}\rho_\tau x_{j,\tau} \\ &= \mathbb{E}_t^*\left[\rho_{T-1}p_{j,T-1} + \sum_{\tau=t+1}^{T-1}\rho_\tau x_{j,\tau}\right] + \sum_{\tau=1}^{t}\rho_\tau x_{j,\tau} \end{aligned}$$

continuing iteratively,

$$\begin{aligned} &= \rho_t p_{j,t} + \sum_{\tau=1}^{t}\rho_\tau x_{j,\tau} \\ &= d_{j,t} \end{aligned}$$

This is the martingale property. □

Exercise (I.3)**.** Consider security markets with I agents. All agents' preferences have expected utility representations with differentiable, strictly increasing utilities and with common probabilities of states π. Agents are strictly risk averse. The aggregate endowment at date 1 is risk free. (You may assume that there is no consumption at date 0).

Show that fair pricing holds in an equilibrium if security markets are complete, that is, show that $\mathbb{E}_\pi[R^j] = R^f$, where R^j denotes the return on any security j and R^f is the risk-free return.

Solution. Since the utilities are differentiable, we can take the first order conditions. These become,

$$R^f \mathbb{E}[v_i'(c_i)] = \mathbb{E}[v_i'(c_i)R^j]$$

for every consumer i and every security j. Since markets are complete, let us consider an Arrow security that pays one unit in state $\hat{s}$.

$$R^f = \frac{\pi(\hat{s})}{p_{\hat{s}}} \frac{v_i'(c_i(\hat{s}))}{\mathbb{E}[v_i(c_i)]}$$

This must hold for every agent. So,

$$\frac{v_i'(c_i(\hat{s}))}{\mathbb{E}[v_i'(c_i)]} = \frac{v_k'(c_k(\hat{s}))}{\mathbb{E}[v_k'(c_k)]}.$$

This holds for every state. The denominators do not depend on the state. This means we can rewrite the equation as

$$v_i'(c_i(s)) = a_{i,k} v_k'(c_k(s)).$$

Since every agent has strictly increasing utility and is strictly risk averse, we get the following implications.

$$\begin{aligned} &c_i(\hat{s}) > c_i(\tilde{s}) \\ \Rightarrow &v_i'(c_i(\hat{s})) < v_i'(c_i(\tilde{s})) \\ \Rightarrow &v_k'(c_k(\hat{s})) < v_k'(c_k(\tilde{s})) \\ \Rightarrow &c_k(\hat{s}) > c_k(\tilde{s}) \end{aligned}$$

In other words, every agent's consumption is strongly comonotone with every other agent. Now since the aggregate endowment is risk-free, it must be that each individual consumption is also risk-free. This implies that $v_i'(c_i(s)) = \mathbb{E}[v_i'(c_i)]$. So,

$$R^f = \mathbb{E}\left[\frac{v_i'(c_i)}{\mathbb{E}[v_i'(c)]} R^j\right] = \mathbb{E}[R^j].$$

□

Exercise (I.4)**.** Consider a setting in which financial assets pay normally distributed returns. The representative agent has CARA utility that is represented by the utility aggregator

$$V(c) = -\mathbb{E}[\exp(-ac)] \tag{1}$$

where $a > 0$ is the agent's absolute risk aversion coefficient and $\mathbb{E}[\cdot]$ is the standard expectations operator under the physical probability measure. The agent can trade in a risk-free asset that pays a constant return R^f and a vector of risky assets with returns $R = (R_1, R_2, \ldots, R_N)'$ that are normally distributed with $N(\mathbb{E}[R], \Sigma)$.

The agent holds an initial position $e^f \in \mathbb{R}$ in the risk-free asset and a vector $e \in \mathbb{R}^N$ of the risky assets. The agent can choose a portfolio, consisting of a position θ^f in the risk-free asset, and $\theta \in \mathbb{R}^N$ in the risky assets. After the uncertain state is realized, the agent consumes

$$c = \theta^f R^f + \theta' R. \tag{2}$$

Show that

$$\mathbb{E}[R] - R^f = aCov(R, R^m), \tag{3}$$

where $R^m \equiv e^f R^f + e'R$ is the market portfoilo.

Solution. First substitute equation (2) into equation (1) and look at the agent's maximization problem,

$$\max_{\theta \in \mathbb{R}^N} -\mathbb{E}[\exp(-aR^f(W - \underline{1}'\theta) - a\theta' R)]$$

where W is the initial wealth from the agent's endowment, but will be entirely unimportant for this problem.

Since R is normally distributed, we can rewrite the problem as

$$\max_{\theta} - \exp\left(-aR^f(W - \underline{1}'\theta) - a\theta'\mathbb{E}[R] + \frac{a^2}{2}\theta'\Sigma\theta\right).$$

The first order conditions then give the following.

$$\begin{aligned} &\left(-aR^f + a\mathbb{E}[R] - a^2\Sigma\theta\right) e^{\mathbb{E}[c] + \frac{1}{2}V[c]} = 0 \\ \Rightarrow \quad &\mathbb{E}[R] - R^f = a\Sigma\theta \end{aligned}$$

This is a vector of equalities. Take some security i.

$$\begin{aligned} \mathbb{E}[R^i] - R^f &= a\sum_{j=1}^{N} \theta_j \sigma_{ij} \quad \text{where } \sigma_{ij} \text{ is the covariance of } R^i \text{ with } R^j \\ &= a\sum_{j=1}^{N} e_j \sigma_{ij} \quad \text{by market clearing} \\ &= aCov(R^i, e'R) \quad \text{by linearity of covariance operators} \\ &= aCov(R^i, e^f R^f + e'R) \quad \text{since the risk free security has no variance} \\ &= aCov(R^i, R^m) \end{aligned}$$

This gives us equation (3). ☐

Section II

Question II.1

Consider security markets with infinite time-horizon and uncertainty described by an event tree with a finite number of events at every date $t = 0, 1, \ldots$. There are J securities with dividends $x_j(s_t) \geq 0$ for every j and every event s_t at date $t \geq 1$. There are I agents whose preferences over infinite-time consumption plans are described by discounted time-seperable expected utilities with strictly increasing period-utility functions. (You may assume that discount factors and probabilities are common to all agents.) Agents have consumption endowments ω_t^i for all $t \geq 0$ and initial portfolios of securities $\hat{h}_0^i \in \mathbb{R}_+^J$. Consumption is restricted to be positive.

Exercise (i)**.** State definitions of natural debt bounds (where debt cannot exceed discounted present value of future endowments) and equilibrium under natural debt constraints.

Solution.

Definition. Let $q(s_t)$ be the event prices. Then the **natural debt bounds** are

$$N^i(s_t) \equiv \frac{1}{q(s_t)} \sum_{\tau=t}^{\infty} \sum_{s_\tau \subset s_t} q(s_\tau)\omega^i(s_\tau).$$

Definition. An **equilibrium under natural debt constraints** is a price process p and consumption-portfolio allocation $\{c^i, h^i\}_{i=1}^I$ such that the consumption plan c^i and the portfolio strategy h^i are a solution to agent i's choice problem,

$$\begin{aligned}
\max_{c,h} \quad & u(c) \\
\text{s.t.} \quad & c(s_0) + p(s_0)h(s_0) = \omega^i(s_0) + p(s_0)\hat{h}_0^i \\
& c(s_t) + p(s_t)h(s_t) = \omega^i(s_t) + [p(s_t) + x(s_t)]h(s_t^-) \quad \forall t, s_t \\
& [p(s_{t+1}) + x(s_{t+1})]h(s_t) \geq -\frac{1}{q(s_{t+1})} \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \subset s_{t+1}} q(s_\tau)\omega^i(s_\tau) \quad \forall t, s_t,
\end{aligned}$$

and markets clear

$$\sum_{i=1}^{I} h^i(s_t) = \bar{h}_0 \quad \forall t, s_t$$

$$\sum_{i=1}^{I} c^i(s_t) = \overline{\omega}(s_t) + x(s_t)\bar{h}_0 \quad \forall t, s_t.$$

□

Exercise (ii)**.** Show that natural debt bounds are not too tight (so that the self-enforcing condition holds with equality) when the punishment for default on debt repayment is zero consumption forever from the default date on.

Solution. Natural debt bounds are **not too tight** if

$$U^{*i}_{s_t}(p, D, -D(s_t)) = \overline{V}^i_d(s_t)$$

where the term on the left is the maximum continuation utility under price system p and debt bounds D when you have initial wealth $-D(s_t)$, and the term on the right is the utility obtained if you default. In this case, the right hand side is the utility of zero consumption forever.

Since we have initial wealth of $-D(s_t)$, this period's budget constraint is

$$c(s_t) + p(s_t)h(s_t) \leq -D(s_t).$$

The current period's debt constraint is

$$\begin{aligned} [p(s_{t+1}) + x(s_{t+1})]h(s_t) &\geq -D(s_{t+1}) \\ &= -\frac{1}{q(s_{t+1})} \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \subset s_{t+1}} q(s_\tau)\omega^i(s_\tau). \end{aligned}$$

Remember that from the definition of event prices, we have

$$p(s_t) = \frac{1}{q(s_t)} \sum_{s_{t+1} \subset s_t} q(s_{t+1})[p(s_{t+1}) + x(s_{t+1})].$$

This implies the following.

$$\begin{aligned} -D(s_t) &\geq c(s_t) + p(s_t)h(s_t) \\ &= \frac{1}{q(s_t)} \sum_{s_{t+1} \subset s_t} q(s_{t+1})[p(s_{t+1}) + x(s_{t+1})]h(s_t) \\ &\geq c(s_t) - \frac{1}{q(s_t)} \sum_{s_{t+1} \subset s_t} q(s_{t+1}) \frac{1}{q(s_{t+1})} \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \subset s_{t+1}} q(s_\tau)\omega(s_\tau) \\ &= c(s_t) - \frac{1}{q(s_t)} \sum_{\tau=t}^{\infty} \sum_{s_\tau \subset s_t} q(s_\tau)\omega(s_\tau) \\ &= c(s_t) - D(s_t) \end{aligned}$$

This implies that we must have zero consumption this period, and that we will be exactly on the debt constraint next period as well. Continuing the logic, this implies that the only consumption stream in the budget set is zero consumption forever. Thus, the maximized value going forward is the utility of zero consumption forever. This means that the natural debt bounds are not too tight when the punishment of default is zero consumption forever. □

Exercise (iii)**.** Show that every consumption allocation in equilibrium under natural debt constraints in dynamically complete markets and with zero price bubbles is Pareto optimal.

Solution. Start with any allocation that satisfies the security markets budget constraints with equality.

$$c(s_0) + p(s_0)h(s_0) = \omega(s_0) + p(s_0)\hat{h}(s_0)$$

$$c(s_t) + p(s_t)h(s_t) = \omega(s_t) + [p(s_t) + x(s_t)]h(s_{t-1})$$

Let us redefine the endowment process by $\hat{\omega}(s_0) = \omega(s_0) + p(s_0)\hat{h}(s_0)$ and $\hat{\omega}(s_t) = \omega(s_t)$ otherwise. This implies the following.

$$\begin{aligned} c(s_0) - \hat{\omega}(s_0) &= -p(s_0)h(s_0) \\ &= -\frac{1}{q(s_0)} \sum_{s_1 \subset s_0} q(s_1)[p(s_1) + x(s_1)]h(s_0) \\ &= 1\frac{1}{q(s_0)} \sum_{s_1 \subset s_0} q(s_1)[c(s_1) - \hat{\omega}(s_1) + p(s_1)h(s_1)] \end{aligned}$$

We can rearrange the equation.

$$\sum_{t=0}^{1} \sum_{s_t} q(s_t)(c(s_t) - \hat{\omega}(s_t)) = -\sum_{s_1 \subset s_0} q(s_1)p(s_1)h(s_1)$$

Continue the same process.

$$\sum_{t=0}^{\infty} \sum_{s_t} q(s_t)(c(s_t) - \hat{\omega}(s_t)) = -\lim_{T\to\infty} \sum_{s_T \subset s_0} q(s_T)p(s_T)h(s_T)$$

The price bubble on an asset is defined by $\sigma = \lim_{T\to\infty} q(s_T)p(s_T)$. We assumed in the problem that this was equal to zero. Since we have natural debt bounds, we also know that $h(s_t)$ is bounded. This means that the right hand side of the above equation must be zero. Therefore,

$$\sum_{t=0}^{\infty} q(s_t)c(s_t) \leq \sum_{t=0}^{\infty} q(s_t)\hat{\omega}(s_t).$$

This is the budget constraint in an Arrow-Debreu problem. This shows that the consumption stream from any allocation in the security markets setup that satisfies the budget constraints with equality is also feasible in the Arrow-Debreu setup.

Now take any consumption stream that satisfies the Arrow-Debreu budget constraint with equality. Since markets are dynamically complete, for any s_t there exists a portfolio $h(s_t)$ such that

$$q(s_{t+1})[p(s_{t+1}) + x(s_{t+1})]h(s_t) = \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \subset s_{t+1}} q(s_\tau)(c(s_\tau) - \hat{\omega}(s_\tau)).$$

We can use that equation for every state to get a portfolio strategy $h(s_t)$ such that $\{c(s_t), h(s_t)\}$ will satisfy the security market budget constraints. This means that any allocation satisfying the Arrow-Debreu budget constraint also can satisfy the security markets budgets constraints with some portfolio strategy.

Since utilities are strictly increasing, we only need to worry about the allocations that make the budget constraints hold with equality.

Since the budget feasible sets are the same in the two setups, the optimal consumption stream must also be the same for the right endowments. We already know that a competitive equilibrium in the Arrow-Debreu setup is Pareto optimal. This shows that a competitive equilibrium in the security markets must also be Pareto optimal, because it is an AD-CE. $\square$

Question II.2

An agent has Epstein-Zin preferences with value recursion

$$U_t = \left[(1-\beta)(C_t)^{1-\rho} + \beta[\mathcal{R}_t(U_{t+1})]^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

where $\mathcal{R}_t(U_{t+1})$ represents the risk adjustment of the continuation value

$$\mathcal{R}_t(U_{t+1}) = \left(\mathbb{E}\left[(U_{t+1})^{1-\gamma} \mid \mathcal{F}_t\right]\right)^{\frac{1}{1-\gamma}}.$$

Assume that the intertemporal elasticity of substitution, $\rho = 1$ and that growth rate of consumption can potentially take two values $g \in \{g_1, g_2\}$ every period.

Exercise (a)**.** Scenario 1: Assume that the growth rate is drawn from a distribution $(p, 1-p)$ initially, and then held fixed over time. Derive the expression for $v \equiv \log\left(\frac{U}{C}\right)$ (before the growth rate is known) in terms of the primitives, i.e. γ, β, and the distribution of g.

Solution. First recall that as ρ goes to 1, CES preferences become Cobb-Douglas.

$$U_t = C_t^{1-\beta}\mathcal{R}_t(U_{t+1})^{\beta}$$

We can now write thte ratio of interest easily then recursively substitute in.

$$\begin{aligned}
\frac{U_t}{C_t} &= \left[\mathcal{R}_t\left(\frac{U_{t+1}}{C_t}\right)\right]^{\beta} \\
&= \left[\mathcal{R}_t\left(\frac{U_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)\right]^{\beta} \\
&= \left[\mathcal{R}_t\left(\left[\mathcal{R}_{t+1}\left(\frac{U_{t+2}}{C_{t+2}}\frac{C_{t+2}}{C_{t+1}}\right)\right]^{\beta}\frac{C_{t+1}}{C_t}\right)\right]^{\beta} \\
&\vdots \\
&= \left[\mathcal{R}_t\left(\left[\mathcal{R}_{t+1}\left(\left[\mathcal{R}_{t+2}\left(\cdots\frac{C_{t+3}}{C_{t+2}}\right)\right]^{\beta}\frac{C_{t+2}}{C_{t+1}}\right)\right]^{\beta}\frac{C_{t+1}}{C_t}\right)\right]^{\beta} \\
&= \mathbb{E}_t\left[\left(\left[\mathcal{R}_{t+1}\left(\left[\mathcal{R}_{t+2}\left(\cdots\frac{C_{t+3}}{C_{t+2}}\right)\right]^{\beta}\frac{C_{t+2}}{C_{t+1}}\right)\right]^{\beta}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{\frac{\beta}{1-\gamma}}
\end{aligned}$$

This was done without any assumption on the distribution of g. Now we use the fact that g is picked in the first period then, held fixed forever. This means that $\mathcal{R}$ isn't doing anything from $t+1$ on. Let I denote everything in the above equation inside the first parenthesis and suppose g_i was drawn.

$$\begin{aligned} I &= \frac{C_{t+1}}{C_t}\left(\frac{C_{t+2}}{C_{t+1}}\left(\frac{C_{t+3}}{C_{t+2}}\cdots\right)^{\beta}\right)^{\beta} \\ &= \prod_{\tau=0}^{\infty}\left(\frac{C_{t+\tau}}{C_t}\right)^{\beta^{\tau}} \\ \Rightarrow \quad \log(I) &= \sum_{\tau=0}^{\infty}\beta^{\tau}\log\left(\frac{C_{t+\tau}}{C_t}\right) \\ &= \frac{g_i}{1-\beta} \end{aligned}$$

We can now plug this back into the above equation.

$$\begin{aligned} \frac{U_t}{C_t} &= \left(pe^{\frac{1-\gamma}{1-\beta}g_1} + (1-p)e^{\frac{1-\gamma}{1-\beta}g_2}\right)^{\frac{\beta}{1-\gamma}} \\ \Rightarrow \quad v_t = \log\left(\frac{U_t}{C_t}\right) &= \frac{\beta}{1-\beta}\frac{1-\beta}{1-\gamma}\log\left(pe^{\frac{1-\gamma}{1-\beta}g_1} + (1-p)e^{\frac{1-\gamma}{1-\beta}g_2}\right) \end{aligned}$$

I know that the $1-\beta$ terms cancel, but if we leave them in, it will look cool later. □

Exercise (b)**.** Scenario 2: Now suppose that the growth rate is drawn every period with probabilities $(p, 1-p)$. Again derive the expression for $\tilde{v} \equiv \log(\frac{U}{C})$ in terms of the primitives as before.

Solution. It starts exactly the same way.

$$\begin{aligned} \frac{U_t}{C_t} &= \left[\mathcal{R}_t\left(\left[\mathcal{R}_{t+1}\left(\left[\mathcal{R}_{t+2}\left(\cdots\frac{C_{t+3}}{C_{t+2}}\right)\right]^{\beta}\frac{C_{t+2}}{C_{t+1}}\right)\right]^{\beta}\frac{C_{t+1}}{C_t}\right)\right]^{\beta} \\ &= \mathbb{E}_t\left[\left(\mathbb{E}_{t+1}\left[\left(\mathbb{E}_{t+2}\left[\left(\cdots\frac{C_{t+3}}{C_{t+2}}\right)^{1-\gamma}\right]^{\frac{\beta}{1-\gamma}}\frac{C_{t+2}}{C_{t+1}}\right)^{1-\gamma}\right]^{\frac{\beta}{1-\gamma}}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{\frac{\beta}{1-\gamma}} \end{aligned}$$

Now since we assumed the growth was iid, we can seperate the expectations and use the law of iterated expectations to get a simpler expression.

$$\begin{aligned}
\frac{U_t}{C_t} &= \mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{\frac{\beta}{1-\gamma}} \mathbb{E}_t\left[\left(\frac{C_{t+2}}{C_{t+1}}\right)^{1-\gamma}\right]^{\frac{\beta^2}{1-\gamma}} \mathbb{E}_t\left[\left(\frac{C_{t+2}}{C_{t+1}}\right)^{1-\gamma}\right]^{\frac{\beta^3}{1-\gamma}} \dots \\
&= \left(\prod_{\tau=0}^{\infty}\left(pe^{(1-\gamma)g_1} + (1-p)e^{(1-\gamma)g_2}\right)^{\beta^\tau}\right)^{\frac{\beta}{1-\gamma}} \\
\Rightarrow \quad \log\left(\frac{U_t}{C_t}\right) &= \frac{\beta}{1-\gamma}\sum_{\tau=0}^{\infty}\beta^\tau \log\left(pe^{(1-\gamma)g_1} + (1-p)e^{(1-\gamma)g_2}\right) \\
\Rightarrow \quad \tilde{v} &= \frac{\beta}{1-\beta}\frac{1}{1-\gamma}\log\left(pe^{(1-\gamma)g_1} + (1-p)e^{(1-\gamma)g_2}\right)
\end{aligned}$$

□

Exercise (c)**.** Finally show that $\gamma > 1$ is necessary and sufficient for the agent to prefer the consumption process where the consumption level is drawn every period.

Solution. We know that when $\gamma = 1$ we have expected utility and the agent is indifferent between the two options. Now define

$$\Omega(\theta) = \frac{\beta}{1-\beta}\frac{1}{\theta}\log\left(\mathbb{E}_t[e^{\theta g}]\right).$$

Notice that $v = \Omega\left(\frac{1-\gamma}{1-\beta}\right)$ and $\tilde{v} = \Omega(1-\gamma)$. We want to compare v and $\tilde{v}$. We know that $\frac{1-\gamma}{1-\beta} > 1-\gamma$ if and only if $\gamma < 1$. All we need to know now is whether Ω is an increasing or decreasing function. We simply differentiate with respect to θ.

$$\Omega'(\theta) = -\frac{\beta}{1-\beta}\frac{1}{\theta^2}\log\left(\mathbb{E}\left[e^{\theta g}\right]\right) + \frac{\beta}{1-\beta}\frac{1}{\theta}\frac{\mathbb{E}\left[ge^{\theta g}\right]}{\mathbb{E}\left[e^{\theta g}\right]}$$

It's not immediately obvious whether this is positive or negative. We're going to use a funny little trick to show the sign of this function. First let

$$m = \frac{e^{\theta g}}{\mathbb{E}\left[e^{\theta g}\right]}.$$

It's clear that $m \geq 0$ and that $\mathbb{E}[m] = 1$. This implies that $E[m\log(m)] \geq 0$ (see the exercise in section 1). This gives us the following.

$$\begin{aligned}
\mathbb{E}\left[\frac{e^{\theta g}}{\mathbb{E}[e^{\theta g}]}\log\left(\frac{e^{\theta g}}{\mathbb{E}[e^{\theta g}]}\right)\right] &\geq 0 \\
\Rightarrow \quad \mathbb{E}\left[\frac{e^{\theta g}\log(e^{\theta g})}{\mathbb{E}[e^{\theta g}]}\right] - \log\left(\mathbb{E}[e^{\theta g}]\right) &\geq 0 \\
\Rightarrow \quad \frac{\beta}{1-\beta}\frac{1}{\theta}\mathbb{E}\left[\frac{ge^{\theta g}}{\mathbb{E}[e^{\theta g}]}\right] - \frac{\beta}{1-\beta}\frac{1}{\theta^2}\log\left(\mathbb{E}[e^{\theta g}]\right) &\geq 0 \\
\Rightarrow \quad \Omega'(\theta) &\geq 0
\end{aligned}$$

Since Ω is increasing, we have proved that $v > \tilde{v}$ if and only if $\gamma < 1$. The iid consumption process is prefered if and only if $\gamma > 1$. $\square$

Question II.3

An aggregate endowment of one unit is split between two agents, 1 and 2 as follows:

$$y_1(s) + y_2(s) = 1,$$

where $s \in \{h, \ell\}$ and the endowment of agent 1 satisfies $y_1(h) > \frac{1}{2}$ and $y_1(\ell) < \frac{1}{2}$. Both types of agents can trade a complete set of arrow securities that are in zero net supply. Their preferences over consumption streams are given by

$$\mathbb{E}^i \left[\sum_{t=0}^{\infty} \beta^t \log(c_i(s^t)) \right],$$

where $\mathbb{E}^i$ is an expectation operator under the probability measure that makes s_t i.i.d. with $p^i(h) = p^i \in (0, 1)$. Let $p^0(h) = \frac{1}{2}$ be the probability measure that nature uses to draw s_t. Histories are represented with $s^t = (s_0, s_1, \ldots, s_t)$.

Exercise (a)**.** Define a competitive equilibrium –allocation, prices, etc– for this economy.

Solution. □

Exercise (b)**.** Let $q_t(s^{t+\tau})$ be the price of one unit of consumption in state $s^{t+\tau}$ at history s^t and $\omega_{i,t}(s^t) = \sum_{\tau}^{\infty} q_t(s^{t+\tau}) y_i(s^{t+\tau})$ be agent i's wealth, or the present discounted value of his endowment stream. Show that the wealth dynamics are given by

$$R_t(s^t) \equiv \frac{\omega_{2,t}(s^t)}{\omega_{1,t}(s^t)} = \left(\frac{p^2(s_t)}{p^1(s_t)} \right) \left(\frac{\omega_{2,t-1}(s^{t-1})}{\omega_{1,t-1}(s^{t-1})} \right).$$

Solution. □

Exercise (c)**.** Suppose $p^1 = p^0 = \frac{1}{2}$ and $p^2 \neq \frac{1}{2}$. Use the law of motion in the previous step to show that $\lim_{t \to \infty} c_t^i = 1$. (Hint: Use the Law of Large Numbers to approximate the limiting behavior of R_t and then use the optimal consumption rules).

Solution. □

Question II.4

Consider the following variation of the model of Kyle (1985) with singe trading date: There is a single risky security whose future payoff, denotes by $\tilde{v}$, is normally distributed with mean $\bar{v}$ and variance $\sigma_v^2 > 0$. There are two strategic, informed traders, a market maker, and liquidity traders. The demand of liquidity traders, denoted by $\tilde{z}$, is normally distributed with zero mean and variance $\sigma_z^2 > 0$, independent of $\tilde{v}$. Neither the strategic traders not the market maker can observe liquidity demand.

Prior to trading, strategic trader 1 observes private signal θ_1 with trader 2 observes private signal θ_2. Those signals are realizations of random variables $\tilde{\theta}_1$ and $\tilde{\theta}_2$, respectively, where

$$\tilde{\theta}_1 = \tilde{v} + \tilde{\epsilon} \quad \text{and} \quad \tilde{\theta}_2 = \tilde{\epsilon},$$

and $\tilde{\epsilon}$ is normally distributed with zero mean and variance $\sigma_\epsilon^2 > 0$, independent of $(\tilde{v}, \tilde{z})$. Strategic traders and the market maker are risk neutral.

Exercise (i)**.** State a definition of a linear equilibrium.

Solution. A linear equilibrium consists of linear functions $x_1(\theta_1)$, $x_2(\theta_2)$, and $P(y)$ such that

$$x_1(\theta_1) = \operatorname*{argmax}_x \mathbb{E}\left[\ x\left(\tilde{v} - P(x + x_2(\tilde{\theta}_2) + \tilde{z})\right)\ \mid \theta_1\ \right], \tag{4}$$

$$x_2(\theta_2) = \operatorname*{argmax}_x \mathbb{E}\left[\ x\left(\tilde{v} - P(x + x_1(\tilde{\theta}_1) + \tilde{z})\right)\ \mid \theta_2\ \right], \tag{5}$$

and

$$P(y) = \mathbb{E}[\ \tilde{v} \mid y = x_1(\tilde{\theta}_1) + x_2(\tilde{\theta}_2) + \tilde{z}\ ]. \tag{6}$$

Since these functions are linear, we will write them as $x_1(\theta_1) = \alpha_0 + \alpha_1\theta_1$, and $x_2(\theta_2) = \beta_0 + \beta_1\theta_2$, and $P(y) = \lambda_0 + \lambda_1 y$. □

Exercise (ii)**.** Assume that $\mathbb{E}[\tilde{v}] = 0$. Find a linear equilibrium.

Solution. Start with agent 1's maximization problem.

$$\begin{aligned}
& \max_x \ \mathbb{E}\left[\ x\left(\tilde{v} - P(x + x_2(\tilde{\theta}_2) + \tilde{z})\right)\ \mid \theta_1\ \right] \\
= & \max_x \ \mathbb{E}\left[\ x\left(\tilde{v} - \lambda_0 - \lambda_1(x + \beta_0 + \beta_1\tilde{\theta}_2 + \tilde{z})\right)\ \mid \theta_1\ \right] \\
= & \max_x \ \mathbb{E}\left[\ \tilde{v}x - \lambda_0 x - \lambda_1 x^2 - \lambda_1\beta_0 x - \beta_1\lambda_1\tilde{\theta}_2 x - \lambda_1\tilde{z}x \mid \theta_1\ \right] \\
= & \max_x \ \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\epsilon^2}\theta_1 x - \lambda_0 x - \lambda_1 x^2 - \lambda_1\beta_0 x - \frac{\sigma_\epsilon^2}{\sigma_v^2 + \sigma_\epsilon^2}\theta_1\beta_1\lambda_1 x
\end{aligned}$$

The first order condition for this is

$$2\lambda_1 x^* = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\epsilon^2}\theta_1 - \lambda_0 - \lambda_1\beta_0 - \frac{\sigma_\epsilon^2}{\sigma_v^2 + \sigma_\epsilon^2}\theta_1\beta_1\lambda_1.$$

We now have that $x_1(\theta_1) = \alpha_0 + \alpha_1\theta_1$ where

$$\alpha_0 = -\frac{\lambda_0 + \lambda_1\beta_0}{1\lambda_1},$$

and

$$\alpha_1 = \frac{\frac{\sigma_v^2}{\sigma_v^2+\sigma_\epsilon^2} - \frac{\sigma_\epsilon^2}{\sigma_v^2+\sigma_\epsilon^2}\beta_1\lambda_1}{2\lambda_1}.$$

Now we can solve agent two's problem in the same way.

$$\begin{aligned}
& \max_x \ \mathbb{E}\left[\ x\left(\tilde{v} - P(x + x_1(\tilde{\theta}_1) + \tilde{z})\right)\ \mid \theta_2\ \right] \\
= & \max_x \ \mathbb{E}\left[\ x\left(\tilde{v} - \lambda_0 - \lambda_1(x + \alpha_0 + \alpha_1\tilde{\theta}_1 + \tilde{z})\right)\ \mid \theta_2\ \right] \\
= & \max_x \ \mathbb{E}\left[\ \tilde{v}x - \lambda_0 x - \lambda_1 x^2 - \lambda_1\alpha_0 x - \alpha_1\lambda_1\tilde{\theta}_1 x - \lambda_1\tilde{z}x \mid \theta_2\ \right] \\
= & \max_x \ -\lambda_0 x - \lambda_1 x^2 - \alpha_0\lambda_1 x - \alpha_1\lambda_1\theta_2 x
\end{aligned}$$

The first order condition for this is

$$2\lambda_1 x^* = -\lambda_0 - \alpha_0\lambda_1 - \alpha_1\lambda_1\theta_0.$$

We now have that $x_2(\theta_2) = \beta_0 + \beta_1\theta_2$ where

$$\beta_0 = -\frac{\lambda_0 + \alpha_0\lambda_1}{2\lambda_1},$$

and

$$\beta_1 = -\frac{\alpha_1}{2}.$$

Now we can look at the market maker's problem.

$$\mathbb{E}[\ \tilde{v} \mid y = \tilde{x}_1 + \tilde{x}_2 + \tilde{z}\ ] = \mathbb{E}\left[\ \tilde{v} \mid y = \alpha_0 + \alpha_1\tilde{\theta}_1 + \beta_0 + \beta_1\tilde{\theta}_2 + \tilde{z}\ \right]$$

Notice that

$$y = \alpha_0 + \beta_0 + \alpha_1\tilde{v} + (\alpha_1 + \beta_1)\tilde{\epsilon} + \tilde{z}.$$

For convenience we will define

$$\hat{y} = \frac{y - \alpha_0 - \beta_0}{\alpha_1} = \tilde{v} + \left(1 + \frac{\beta_1}{\alpha_1}\right)\tilde{\epsilon} + \frac{1}{\alpha_1}\tilde{z}.$$

We can now do the expectation more easily.

$$\mathbb{E}[\ \tilde{v} \mid \hat{y}\ ] = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\epsilon^2(1 + \frac{\beta_1}{\alpha_1})^2 + \frac{\sigma_z^2}{\alpha_1^2}}\hat{y}$$

Plugging in for $\hat{y}$ gives us our linear equation for $P(y)$.

This system of equations can be simplified somewhat. First conjecture notice that $\alpha_0 = \beta_0 = \lambda_0 = 0$. Then it becomes,

$$\beta_1 = -\frac{\alpha_1}{2},$$

$$\lambda_1 = \frac{\sigma_v^2}{\sigma_v^2 + \frac{\sigma_\epsilon^2}{4} + \frac{\sigma_z^2}{\alpha_1^2}} \frac{1}{\alpha_1},$$

and

$$\alpha_1 = \frac{\frac{\sigma_v^2}{\sigma_v^2+\sigma_\epsilon^2} - \frac{\sigma_\epsilon^2}{\sigma_v^2+\sigma_\epsilon^2}\beta_1\lambda_1}{2\lambda_1}.$$

We can plug the first two equations into the third one and solve for α_1.

$$\alpha_1 = \frac{\frac{\sigma_v^2}{\sigma_v^2+\sigma_\epsilon^2} + \frac{1}{2}\frac{\sigma_\epsilon^2}{\sigma_v^2+\sigma_\epsilon^2}\frac{\sigma_v^2}{\sigma_v^2+\frac{1}{4}\sigma_\epsilon^2+\frac{\sigma_z^2}{\alpha_1^2}}}{2\frac{\sigma_v^2}{\sigma_v^2+\frac{1}{4}\sigma_\epsilon^2+\frac{\sigma_z^2}{\alpha_1^2}}\frac{1}{\alpha_1}}$$

Divide both sides by alpha and bring the denomenator up to the top.

$$\begin{aligned}
1 &= \frac{1}{2}\frac{\sigma_v^2 + \frac{1}{4}\sigma_\epsilon^2 + \frac{\sigma_z^2}{\alpha_1^2}}{\sigma_v^2+\sigma_\epsilon^2} + \frac{1}{4}\frac{\sigma_\epsilon^2}{\sigma_v^2+\sigma_\epsilon^2} \\
&= \frac{1}{2}\frac{\sigma_v^2 + \frac{1}{4}\sigma_\epsilon^2 + \frac{\sigma_z^2}{\alpha_1^2}}{\sigma_v^2+\sigma_\epsilon^2} \\
\Rightarrow \quad \frac{\sigma_z^2}{\alpha_1^2} &= 2(\sigma_v^2+\sigma_\epsilon^2) - \sigma_v^2 - \frac{3}{4}\sigma_\epsilon^2 \\
\Rightarrow \quad \alpha_1 &= \sqrt{\frac{\sigma_z^2}{\sigma_v^2 + \frac{5}{4}\sigma_\epsilon^2}}
\end{aligned}$$

Notice that this is very similar to the example Jan did in his notes. The answer in his example (translated to this notation) is $\alpha_1 = \sqrt{\frac{\sigma_z^2}{\sigma_v^2}}$. His example in the notes does not have any ϵ error. The insider observes the value perfectly. If we let σ_ϵ^2 go to zero, we get the same answer as in his notes. □

Exercise (iii)**.** Show that trader 2 trades against his signal in equilibrium of part (ii), that is, his demand is positive when the signal is negative and vice versa. Give an intuitive explanation of this result.

Solution. First, α_1 is clearly positive because it is just a square root of positive numbers. Since $\beta_1 = -\frac{\alpha_1}{2}$, it must be negative. β_1 is the response of agent 2 to his signal. Since it is negative, that means that when he gets a positive signal he demands a negative amount, and when he gets a negative signal he demands a positive amount.

The intuition is as follows. When agent 2 gets a positive signal, he knows that agent 1 is overvaluing the security. Agent 1 will then demand too much. This will lead the price to be higher than the true value on average. Then, agent 2 wants to take a short position.

The reverse is true when agent 2 gets a negative signal. □