Financial Prelim 2015 Solutions

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Section I

Exercise (I.1)**.** Consider the optimal portfolio choice problem with one risky security and a risk-free security. The agent has constant absolute risk aversion equal to α. The risky security has normaly distributed payoff with mean μ and variance σ^2. Its price is p. The risk-free security has return r. Derive the agent's optimal investment in the risky security as a function of α, μ, σ, p, and r.

Solution. Set up the problem as choosing the number of shares of each security subject to some initial wealth, W.

$$\begin{aligned} \max_{\theta^f,\theta^r} \quad & -\mathbb{E}[\exp(-\alpha(\theta^f \bar{r} + \theta^r r))] \\ \text{s.t.} \quad & \theta^f + p\theta^r \leq W \end{aligned}$$

Simplifying slightly, the problem becomes

$$\max_{\theta} -\mathbb{E}[\exp(-\alpha\bar{r}(W - p\theta) - \alpha\theta r)].$$

Since the risky asset is normally distributed, this is the same as the following problem.

$$\max_{\theta} - \exp\left(-\alpha\bar{r}(W - p\theta) - \alpha\mu\theta + \alpha^2\frac{\sigma^2}{2}\theta^2\right)$$

The first order condition is

$$\left(-\alpha\bar{r}p + \alpha\mu - \alpha^2\sigma^2\theta^*\right)\exp\left(-\alpha\bar{r}(W - p\theta^*) - \alpha\mu\theta^* + \alpha^2\frac{\sigma^2}{2}\theta^{*2}\right) = 0.$$

We can divide away the exponential term and solve for θ.

$$\theta^* = \frac{1}{\alpha\sigma^2}(\mu - rp)$$

In the above equation θ^* is the number of shares of the risky security. The wealth invested in the risky security is $p\theta^*$. □

Exercise (I.2)**.** Consider a static economy with N securities with a price vector $q \in \mathbb{R}^N$ and payoffs given by an $N \times K$ matrix D. The row D_i thus correspond to the vector of state-dependent payoffs of security i. A portfolio $\theta \in \mathbb{R}^N$ is a vector of holdings of individual securities, with market value $p \cdot \theta$ and payoff $D'\theta \in \mathbb{R}^K$.

1. When is the pair (D, q) said to be arbitrage free?
2. Show that absence of arbitrage is necessary and sufficient for existence of a state-price vector $m \in \mathbb{R}^K_{++}$ such that $q = Dm$.

Solution. The pair (D, q) is said to be arbitrage-free if there does not exist $\theta \in \mathbb{R}^N$ such that $q'\theta \leq 0$ and $D'\theta \geq 0$ with at least one strict inequality (note that the second one is a vector of inequalities and any one of them could be strict).

Let's start with the easy direction of the proof. Suppose that there exist $m \in \mathbb{R}^K_{++}$ such that $q = Dm$. Then for any $\theta \in \mathbb{R}^N$,

$$q'\theta = m'D'\theta.$$

Since m is strictly positive, if $q'\theta$ is strictly negative $D'\theta$ must have at least one negative element. Also, if $D'\theta$ is nonnegative and has at least one strictly positive element, then $q'\theta$ is positive. This proves the necessary part of question 2.

Now let's do the other easy but slightly longer direction of the proof. The idea of the proof is going to be the same as any other proof that gives the existence of a price vector. We use the assumptions to construct a closed convex set, then use the seperating hyperplane theorem to get your price. First we assume no arbitrage. Define the following set in $\mathbb{R} \times \mathbb{R}^K$,

$$\mathcal{K} = \left\{(-q'\theta, D'\theta) \;\middle|\; \forall \theta \in \mathbb{R}^N\right\}.$$

This set is obviously closed and convex. In fact, it is a linear subspace of $\mathbb{R}^{K+1}$.

The no arbitrage condition implies that

$$\mathcal{K} \cap \mathbb{R}^{K+1}_+ = \{0\}.$$

We will take the positive orthant to be the other set in our seperating hyperplane arguement. It is clearly closed and convex. Moreover, it is a cone.

The seperating hyperplane theorem gives us the existence of a vector $m \in \mathbb{R}^{K+1}$ such that $k'm \leq 0$ for all $k \in \mathcal{K}$, $r'm \geq 0$ for all $r \in \mathbb{R}^{K+1}_+$, and the second inequality is strict for all $r \neq 0$. Since $r'm > 0$ for all basis vectors r, it must be that m is strictly positive in all entries.

Remember that $\mathcal{K}$ was a linear subspace. This means that if $k \in \mathcal{K}$ then $-k \in \mathcal{K}$. Thus, $k'm = 0$ for all $k \in \mathcal{K}$. To write it in another way, this says

$$\begin{aligned} & -\theta' q m_1 + \theta' D m_{-1} = 0 \quad \forall \theta \in \mathbb{R}^N \\ \Rightarrow \;& q m_1 = D m_{-1}. \end{aligned}$$

Now define $m^* \in \mathbb{R}^K_{++}$ by $m^* = \frac{m_{-1}}{m_1}$, and we have the result. $q = D'm$. $\square$

Exercise (I.3)**.** For a non-negative random variable m such that $\mathbb{E}[m] = 1$ show that $\mathbb{E}[m \log(m)] \geq 0$.

Solution. First define $\phi(m) = m \log(m)$. Note that $\phi''(m) = \frac{1}{m} > 0$ since m is non-negative. So, ϕ is convex in m. Since ϕ is everywhere convex, it always lies above its first-order Taylor expansion.

$$\begin{aligned} \phi(m) &\geq \phi(1) + \phi'(1)(m-1) \\ &= 1 * \log(1) + (\log(1) + 1)(m-1) \\ &= m - 1 \end{aligned}$$

We can now take the expectation of both sides of the equation to get the result. $\mathbb{E}[m \log(m)] = \mathbb{E}[\phi(m)] \geq \mathbb{E}[m-1] = 0$. □

Exercise (I.4)**.** Consider portfolio choice problem with a risky security with (gross) return r and a risk-free security with return $\bar{r}$. The agent has multiple-prior expected utility function with strictly increasing and convace utility function v and a closed and convex set of probability measures $\mathcal{P}$ on a finite state-space. The agent has date-1 state-dependent endowment $w_1 \in \mathcal{M}$. Prove that if $\bar{r} = \mathbb{E}_P[r]$ for some $P \in \mathcal{P}$, then the optimal portfolio is such that the optimal date-1 consumption (equal to portfolio payoff plus endowment) is risk-free.

Solution. The set-up of the problem is the following.

$$\sup_{\theta} \min_{P \in \mathcal{P}} \mathbb{E}_P \left[v(w_1 + (w_0 - \theta)\bar{r} + \theta r) \right]$$

The min is well defined because $\mathcal{P}$ is compact and $\mathbb{E}_P[v]$ is continuous in P. Since w_1 is in the asset span, we can write $w_1 = a + br$. Then, we can see that the agent can acheive risk-free consumption iff they set $\theta = -b$. Plugging that in, the utility payoff is $v(a + (b + w_0)\bar{r})$.

Let $P^* \in \mathcal{P}$ be the probability measure such that $\mathbb{E}_{P^*}[r] = \bar{r}$. Now consider any strategy θ.

$$\begin{aligned} \min_{P \in \mathcal{P}} \mathbb{E}_P[v(w_1 + (w_0 - \theta)\bar{r} + \theta r)] &\leq \mathbb{E}_{P^*}[v(w_1 + (w_0 - \theta)\bar{r} + \theta r)] \\ &\leq v\left(\mathbb{E}_{P^*}[w_1 + (w_0 - \theta)\bar{r} + \theta r]\right) \\ &= v\left(a + b\mathbb{E}_{P^*}[r] + (w_0 - \theta)\bar{r} + \theta \mathbb{E}_{P^*}[r]]\right) \\ &= v\left(a + (w_0 + b)\bar{r}\right) \end{aligned}$$

This was the payoff from the risk-free consumption strategy. Thus, it is optimal to have risk free date-1 consumption. □

Section II

Question II.1

Suppose that there are $S > 1$ states of nature at date 1. There are I agents whose preferences over state-contingent consumption plans are described by expected utility functions

$$\sum_{s=1}^{S} \pi_s v^i(c_s^i),$$

where c_s^i denotes consumption in state s, π_s is the strictly positive probability of state s (common to all agents), and v^i is the von Neumann-Morgenstern utility. Suppose that utility functions v^i have linear risk tolerance (or, equivalently, hyperbolic risk aversion) with the same slope. There is no consumption at date 0.

The aggregate endowment is $\bar{\omega} = (\bar{\omega}_1, \ldots \bar{\omega}_S)$ with $\bar{\omega}_s > 0$ for every state s and $\bar{\omega}_s \neq \bar{\omega}_{s'}$ for at least one pair of states s, s'.

Exercise (i)**.** State a theorem asserting that agents' consumption plans at every Pareto optimal allocation lie in a two-dimensional subspace of the consumption space $\mathbb{R}^S$. Be as general as you can. Prove the theorem you stated under the additional assumption that utility functions have constant absolute risk aversion.

Solution. He is looking specifically for the following theorem.

Theorem. *If every agent has linear (affine) risk tolerance with the same slope, then each agent's consumption is an affine transformation of the aggregate endowment in any Pareto optimal allocation.*

Proof. I will prove it only for the case of CARA utility. Start with the planner's problem.

$$\max_{\{c_s^i\}_{i,s}} \quad \mathbb{E}\left[\sum_{i=0}^{n} \lambda^i \left(-e^{-\frac{c_s^i}{\alpha^i}}\right)\right]$$
$$\text{s.t.} \quad \sum_{j=0}^{n} c_s^i = \bar{\omega}$$

The first order conditions are

$$\lambda^i \frac{1}{\alpha^i} e^{-\frac{c_s^i}{\alpha^i}} = \frac{\mu_s}{\pi_s}$$

for all agents i and states s, where μ_s are the multipliers on the constraints and

π_s. This implies the following.

$$\begin{aligned}
\lambda^i \frac{1}{\alpha^i} e^{-\frac{c_s^i}{\alpha^i}} &= \lambda^j \frac{1}{\alpha^j} e^{-\frac{c_s^j}{\alpha^j}} \\
\Rightarrow \quad \log\left(\frac{\lambda^i}{\alpha^i}\right) - \frac{c_s^i}{\alpha^i} &= \log\left(\frac{\lambda^j}{\alpha^j}\right) - \frac{c_s^j}{\alpha^j}
\end{aligned}$$

multiply by α^j and sum over all the agents

$$\Rightarrow \quad \sum_{j=0}^{n} \left[\alpha^j \log\left(\frac{\lambda^i}{\alpha^i}\right) - \frac{\alpha^j}{\alpha^i} c_s^i\right] = \sum_{j=0}^{n} \left[\alpha^j \log\left(\frac{\lambda^j}{\alpha^j}\right) - c_s^j\right]$$

$\sum_{j=0}^{n} c^i = \bar{\omega}$, and all the j's sum out and there are no s's other than on the consumption. So you can write,

$$c^i = A^i + B^i \bar{\omega}$$

□

Notice that $A^i + B^i\bar{\omega}$ is a two dimensional subspace of $\mathbb{R}^s$. □

Exercise (ii)**.** Suppose that there are security markets. Markets may be incomplete, but it is assumed that agents' endowments and the risk-free payoff lie in the asset span. Show that, in an equilibrium in security markets, every security whose payoff is co-monotone with the aggregate endowment must have expected return greater than or equal to the risk-free return.

Solution. First recall the following theorem from Jan Werner that I will state without proof.

Theorem. *If x and y are co-monotone, then Cov$(x, y) \geq 0$.*

Now recall from the theorem we stated in the previous part, that $c^i = A^i + B^i\bar{\omega}$ where $\bar{\omega}$ is the aggregate endowment/aggregate consumption. When we proved this for CARA utility above, we got $B^i = \frac{\alpha^i}{\sum_j \alpha^j} > 0$. In the more general proof, we still get that $B^i > 0$. This means that each individual's consumption is (strictly) co-monotone with aggregate consumption.

Now let R be the return on some security that is co-monotone with the aggregate endowment, and thus also co-monotone with each agent's optimal consumption. From the first order conditions, we can get the following.

$$\begin{aligned}
1 &= \mathbb{E}\left[\frac{\partial_1 v(c)}{\partial_0 v(c)} R\right] \\
&= \mathbb{E}\left[\frac{\partial_1 v(c)}{\partial_0 v(c)}\right][R] + \text{Cov}\left(\frac{\partial_1 v(c)}{\partial_0 v(c)}, R\right) \\
&= R^f \mathbb{E}[R] + \frac{\text{Cov}(\partial_1 v(c), R)}{\mathbb{E}[\partial_0 v(c)]}
\end{aligned}$$

$$\Rightarrow \quad \mathbb{E}[R] = R^f - R^f \frac{\mathrm{Cov}(\partial_1 v(c), R)}{\mathbb{E}[\partial_0 v(c)]}$$

We have assumed that v is increasing and concave in c. So, $\mathbb{E}[\partial_0 v(c)]$ is positive and $\partial_1 v(c)$ is negatively co-monotone with c. By the above theorem, this means that $\mathrm{Cov}(-\partial_1 v(c), R) \geq 0$.

Thus, $\mathbb{E}[R] \geq R^f$. $\square$

Question II.2

Let R be a vector of random variables, interpreted as gross returns (i.e., unit cost payoffs) and let ω be a set of weights that add up to one. Assume that R contains a bond with constant returns R^f. Let m be a stochastic discount factor that satisfies the Euler equation,

$$\mathbb{E}[m\omega' R] = 1 \quad \forall \omega : \quad \sum_i \omega_i = 1.$$

Exercise (a)**.** Derive the Hansen Jaggannathan volatility bound for m:

$$\frac{\sigma(m)}{\mathbb{E}[m]} \geq \frac{\|\mathbb{E}[\omega' R - R^f]\|}{\sigma(\omega' R)} \quad \forall \omega : \quad \sum_i \omega_i = 1.$$

Solution. Starting from the Euler equation,

$$\begin{aligned}
1 &= \mathbb{E}[m\omega' R] \\
\Rightarrow \quad 1 &= \mathbb{E}[m]\mathbb{E}[\omega' R] + \mathrm{Cov}(m, \omega' R) \\
\Rightarrow \quad \mathbb{E}[\omega' R] - \frac{1}{\mathbb{E}[m]} &= -\mathrm{Corr}(m, \omega' R)\sigma(m)\sigma(\omega' R)\frac{1}{\mathbb{E}[m]} \\
\Rightarrow \quad \left|\mathbb{E}[\omega' R - R^f]\right| &= |\mathrm{Corr}(m, \omega' R)|\sigma(m)\sigma(\omega' R)\frac{1}{\mathbb{E}[m]} \\
&\leq \sigma(m)\sigma(\omega' R)\frac{1}{\mathbb{E}[m]} \\
\Rightarrow \quad \frac{\sigma(m)}{\mathbb{E}[m]} &\geq \frac{\left|\mathbb{E}[\omega' R - R^f]\right|}{\sigma(\omega' R)}
\end{aligned}$$

□

Exercise (b)**.** Derive the Bansal Lehmann bound for m:

$$-\mathbb{E}[\log(m)] \geq \mathbb{E}[\log(\omega' R)] \quad \forall \omega : \quad \sum_i \omega_i = 1.$$

Can you provide an economic interpretation of this bound? (Hint: Think from a perspective of an agent who has log utility)

Solution. We start from the same Euler equation.

$$\begin{aligned}
1 &= \mathbb{E}[m\omega' R] \\
\Rightarrow \quad 0 &= \log\left(\mathbb{E}[m\omega' R]\right) \\
&\geq \mathbb{E}[\log(m\omega' R)] \\
&= \mathbb{E}[\log(m)] + \mathbb{E}[\log(\omega' R)] \\
\Rightarrow \quad -\mathbb{E}[\log(m)] &\geq \mathbb{E}[\log(\omega' R)]
\end{aligned}$$

For intuition, imagine an investor with log utility. Then,

$$-\mathbb{E}[\log(m)] = \mathbb{E}\left[\log\left(\frac{c_t}{c_{t+1}}\right)\right] = \mathbb{E}\left[\log\left(\frac{c_{t+1}}{c_t}\right)\right].$$

Since $\omega' R$ is a return you can get on your wealth, we'll give the following interpretation. The Bansal Lehmann bound says that your expected growth rate on consumption must be at least as large as your expected growth rate on wealth (if you have log utility). More generally, we just have that the inverse of the expected growth rate of marginal utilities must be as large as the expected growth rate on wealth. □

Exercise (c)**.** Suppose m, R were log normally distributed. Compare the two bounds derived in (a) and (b).

Solution. The Bansal and Lehmann 1997 paper answers exactly this question, but I don't get it at all. □

Question II.3

Consider an aggregate consumption process of the following form

$$\log(c_{t+1}) - \log(c_t) = m + x_t + \sigma_t W_{c,t+1}$$

$$x_{t+1} = a x_t + \varphi \sigma_t W_{x,t+1}$$
$$\sigma_{t+1}^2 = \sigma^2 + \nu(\sigma_t^2 - \sigma^2) + \sigma_w W_{w,t+1}.$$

where $0 < \alpha < 1$ and $W_{c,t}$, $W_{x,t}$, $W_{w,t}$ are standar Gaussian innovations, mutually independent and i.i.d. over time.

Consider the following experiment: An agent with Epstein-Zin preferences,

$$U_t = \left\{(1-\beta)c^{\rho} + \beta \left[\mathbb{E}_t[U_{t+1}^{\alpha}]\right]^{\frac{\rho}{\alpha}}\right\}^{\frac{1}{\rho}},$$

faces the consumption stream described above for $t = 1, 2, \ldots$. In particular note that the riskiness of consumption resolves only gradually over time (c_t, x_t are realized only at time t). How much would he pay at time 0 to have all risk resolved in the next period? Assume intertemporal elasticity of substitution, $IES = 1$,

Exercise (a)**.** Derive an expression for U_0.

Solution. □

Exercise (b)**.** Let U_0^* as the utility stream when all risk is resolved at $t = 1$. Derive an expression for U_0^*. (Hint: Use backward induction logic: First derive U_1^* for an agent who faces deterministic stream of consumption. Then use the Epstein-Zin recursion to compute U_0^* be aggregating utility from c_0 and U_1^*).

Solution. □

Exercise (c)**.** Now compute the $\pi^* = 1 - \frac{U_0}{U_0^*}$ as the *timing premium*. Discuss how it varies with parameters α, β, φ, σ^2, ν.

Solution. □

Question II.4

Consider security markets with infinite time-horizon and uncertainty described by an event tree with a finite number of events at every date $t \geq 1$. There are J infinitely-lived securities with dividends $x_j(s_t) \geq 0$ for every j and every event s_t, every $t \geq 1$. Let $p(s_t)$ denote a vector of security prices in event s_t. There are I agents and a single good available for consumption at every date. Each agent i has strictly increasing preferences over infinite-time event-contingent consumption plans, and event contingent endowment of the good at every date t (denoted by ω_t^i) and an initial portfolio of securities (denoted by $\hat{h}_0^i \in \mathbb{R}_+^J$). Consumption is restricted to be positive. You may assume that agents' preferences have discounted expected-utility representation. Agents' portfolio holdings $h(s_t)$ at s_t are restricted by debt constraints of the form

$$[p(s_{t+1}) + x(s_{t+1})]h(s_t) \geq -D(s_{t+1}), \quad \forall s_{t+1} \subset s_t, \quad \forall t \geq 0,$$

where bounds $\{D(s_{t+1})\}$ are positive, and $s_{t+1} \subset s_t$ indicates that event s_{t+1} is a successor of s_t.

Exercise (i)**.** State a definition of an arbitrage under debt constraints and prove that there does not exist arbitrage if and only if there exists strictly positive event prices. You may use a characterization of no-arbitrage in static two period markets without proof.

Solution.

Definition. An **arbitrage under debt constraints** is a portfolio strategy h such that $p_0 h_0 \leq 0$, $z(h,p)(s_t) \geq 0$ for every event s_t, at least one of the inequalities is strict, and $[p(s_{t+1}) + x(s_{t+1})]h(s_t) \geq 0$.

Lemma. *Security prices exclude arbitrage under debt constraints if and only if they exclude one period arbitrage in every state.*

Proof. The forward direction of the proof is immediate. If there is no arbitrage under debt constraints, there cannot be a one period arbitrage. This is clear because a one period arbitrage is an arbitrage under debt constraints.

Now the reverse direction. Assume there is no one period arbitrage. Take any portfolio strategy h such that $p_0 h_0 \leq 0$, $z(h,p)(s_t) \geq 0$, and $[p(s_t)+x(s_t)]h(s_t) \geq 0$. Note that any potential arbitrage must satisfy these conditions. We must have that $p_0 h_0$ and $[p(s_1) + x(s_1)]h_0$ are zero for all s_1, otherwise h_0 would be a one period arbitrage. Since $z(h,p)(s_1) \geq 0$ we must have that $p(s_1)h(s_1) \leq 0$ and either they're both strict or both equal to zero.

Now we can just repeat the same arguement for the next period. We continue this iteratively through infinity. This shows that no h can be an arbitrage under debt constraints. □

Lemma. *In the two period model there exists strictly positive event prices if and only if security prices exclude arbitrage.*

Proof. Take X to be the payoff matrix in the two period model and p the price vector.

Theorem. *Stiemke's Lemma There does not exist $h \in \mathbb{R}^m$ such that $hX \geq 0$ and $hp \leq 0$ with one strict inequality,*
if and only if there exists $q \in \mathbb{R}^n$ such that $p = Xq$ and $q >> 0$.

In the above theorem, h is an arbitrage and q is the event prices. □

These two lemma's together imply the result. □

Exercise (ii)**.** State a definition of price bubble. State a theorem providing sufficient conditions for price bubbles being zero in equilibrium under debt constraints.

Solution.

Definition. A **price bubble** on security j at s_t is

$$\sigma_j(s_t) = p_j(s_t) - \frac{1}{q(s_t)} \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \subset s_t} q(s_\tau) x_j(s_\tau).$$

Where $p_j(s_t)$ is the price of the security and $q(s_t)$ is the event price.

Definition. Agents exhibit **uniform impatience** with respect to the effective aggregate endowment $\hat{w}$ if there exists γ satisfying $0 \leq \gamma < 1$ such that

$$u^i(c^i_-(s_t), c^i(s_t) + \hat{w}, \gamma c^i_+(s_t)) > u^i(c^i)$$

for every i, every s_t, and every c^i such that $0 \leq c^i \leq \hat{w}$.

Theorem. *Assume that agents' utility functions exhibit uniform impatience. Suppose that $(p, \{c^i, h^i\})$ is an equilibrium in security markets under debt constraints and q is a sequence of strictly positive event prices associated with p. If the present value of the aggregate endowment is finite,*

$$\sum_{t=0}^{\infty} \sum_{s_t \in F_t} q(s_t) \bar{w}(s_t) < \infty$$

then the price bubble is zero for every security that is in strictly positive supply.

□

Exercise (iii)**.** Give an example of an equilibrium with strictly positive price bubbles on a security in strictly positive supply. The security can be zero-dividend security (i.e. "fiat money"). Sketch a proof that the prices and allocations in your example are indeed an equilibrium for the utilities, endowments and debt bounds that you specified.

Solution. The example is two agent's playing pitcher-catcher with a single dollar bill. Imagine the following environment.

There are two agents with the following utility with $0 < \delta < 1$.

$$u^i(c) = \sum_{t=0}^{\infty} \delta^t \log(c_t)$$

There is a zero debt constraint. Endowments are $w_t^0 = H$ and $w_t^1 = L$ if t is even, and $w_t^0 = L$ and $w_t^1 = H$ if t is odd, and $\delta H > L$. There is a single security that pays no dividends. The initial security holdings are $h_0^1 = 1$ and $h_0^0 = 0$.

The following price and consumption streams constitute an equilibrium. Let $\eta = \frac{\delta H - L}{1+\delta}$. $p_t = \eta$ for all t. $c_t^i = H - \eta$ if $w_t^i = H$ and $c_t^i = L + \eta$ if $w_t^i = L$. $h_t^i = 1$ if $w_t^i = H$ and $h_t^i = 0$ if $w_t^i = L$. We can quickly see that this is an equilibrium by looking at the Euler equations. For the agent that is not debt constrained,

$$\frac{\delta^t}{c_t^i} p_t = \frac{\delta^{t+1}}{c_{t+1}^i} p_{t+1}.$$

For the constrained agent, the left side must be larger than the right side. Also, the budget constraint must hold for each agent every period.

$$c_t^i + p_t h_t^i \leq w_t^i + p_t h_{t-1}^i$$

Plugging in the proposed equilibrium values, we can quickly see that they satisfy the conditions. □