Econ 8106

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Contents

1 Defining an Economy and Welfare Theorems

First, let's begin by providing a definition for an economy.[1] An economy may be formalized as the set $\mathcal{E}$ where

$$\mathcal{E} = \{\mathcal{L}, n, (X_i)_{i=1}^n, Y, (\omega_i)_{i=1}^n, (u_i)_{i=1}^n\},$$

where $\mathcal{L}$ represents the l-dimensional commodity space (normally chosen to be a subset of $\mathbb{R}^l$), n is the number of agents within the economy, each with their own consumption set X_i, endowment ω_i and utility function $u_i : X_i \to \mathbb{R}$. Lastly, Y represents the technology set, where we assume that Y has constant returns to scale (CRS), implying that Y is a convex cone.[2] Notice that the technology set and the consumption set are both subsets of the commodity space (i.e. $Y \subseteq \mathcal{L}$ and $X_i \subseteq \mathcal{L}$). Next, we may define an allocation.

Definition 1.1. *The set $\{x, y\}$ constitutes an allocation, given $x \in \mathbb{R}^{ln}$, where x is simply a stacked vector of all the consumption sets. This allocation is deemed feasible provided that the sum of individual consumption vectors is equal to the sum of endowments plus technology set; that is, $\sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i + y$, where inputs are negative-valued and outputs are positive-valued within the technology set. This condition is often called resource feasibility (RF), or the market-clearing condition.*

Example. *Consider a simple economy with 2 agents and a commodity space consisting of labor, capital and a final consumption good. The agents are endowed with labor and capital and do not value leisure. In this economy, we have $\omega_1 = (1, 3, 0)'$ and $\omega_2 = (1, 2, 0)'$ where the labor time and capital are represented in the first and second components, respectively. The technology set is characterized by a single firm, taking labor and capital as inputs to produce the consumption good. An allocation, satisfying resource feasibility might be $x_1 = (0, 0, 4)'$ and $x_2 = (0, 0, 2)'$ with $y = (-2, -5, 6)'$ such that the identity is attained.*

For the models considered here, we will assume that they are characterized by a linear price system, or inner product representation:

$$v(x) = \sum_{i=1}^{l} p_i x_i = p \cdot x,$$

where the value of the consumption bundle, $v(x)$, is a linear functional on the commodity space.[3] On a finite space, any linear functional can have an inner product price system representation.

[1] Thank you to Yuki Yao and Isaac Swift for comments and to Monica Tran Xuan, Sergio Ocampo-Diaz and Keyvan Eslami for previous notes on this material.

[2] That is to say, that for scalars α, β, if $y, y' \in Y$, then $\alpha y + \beta y' \in Y$, as well.

[3] A linear functional v is a function such that $v(\alpha x + \beta y) = \alpha v(x) + \beta v(y)$.

Definition 1.2. *A competitive equilibrium (CE) is an allocation* $\{x, y\} \in \mathbb{R}^{ln}_{+} \times \mathbb{R}^{l}$ *and price system* $p \in \mathbb{R}^{l}_{++}$ *where*

$$\forall i, \qquad x_i \in argmax \quad u_i(x_i)$$
$$s.t. \quad p \cdot x_i \leq p \cdot \omega_i, \quad x_i \in X_i, \text{ and}$$
$$y \in argmax \quad p \cdot y$$
$$s.t. \quad y \in Y$$

and the resource feasibility constraint is met.

Definition 1.3. *An allocation* $\{x, y\}$ *is (Strongly) Pareto optimal if there does not exist a feasible allocation* $\{\hat{x}, \hat{y}\}$ *such that*

$$u_i(\hat{x}_i) \geq u_i(x_i) \quad \forall i,$$

and holds with strict inequality for at least one agent i'*.*

Theorem 1.1 (First Welfare Theorem)**.** *Suppose that the number of agents* n *is finite and that agent utility functions are locally non-satiated. Then, any competitive equilibrium allocation is Pareto Optimal.*

Proof. This is a sketch of the proof. Suppose, for a contradiction, there exists a competitive equilibrium which is not Pareto Optimal. Then, by the definition of Pareto optimality, there exists another feasible bundle $\{\hat{x}, \hat{y}\}$ such that for at least one agent i', we observe $u_{i'}(\hat{x}_{i'}) > u_{i'}(x_{i'})$. Given that the allocation $\{x, y\}$ is a competitive equilibrium and that the utility functions are locally non-satiated, each agent fully expends her endowment wealth on consumption. Thus, for agent i',

$$p \cdot \hat{x}_{i'} > p \cdot x_{i'},$$

and for all other agents $i \neq i'$,

$$p \cdot \hat{x}_i \geq p \cdot x_i.$$

In summing over all agents, this leads to $p \cdot \sum_i^n \hat{x}_i > p \cdot \sum_i^n x_i$. Lastly, note that by the definition of competitive equilibrium, firms are maximizing profits; thus, we observe

$$p \cdot \hat{y} \leq p \cdot y.$$

Combining these equations, we have

$$p \cdot (\sum_i^n \hat{x}_i - \hat{y}) > p \cdot (\sum_i^n x_i - y).$$

In a competitive equilibrium, we can prove that the resource constraint will hold with equality. Then we have

$$\sum_i^n \hat{x}_i - \hat{y} \leq \sum_i^n x_i - y = \omega;$$

and multiplying through by the price vector, we have

$$p \cdot (\sum_i^n \hat{x}_i - \hat{y}) \leq p \cdot (\sum_i^n x_i - y) = p \cdot \omega,$$

a contradiction. Therefore, the competitive equilibrium must be a Pareto optimal allocation. □

Theorem 1.2 (Second Welfare Theorem). *Assume that agent utility functions are strictly concave and differentiable. Then, given any Pareto optimal allocation, there exists a price vector p and distribution of endowments* $(\hat{\omega}^i)_{i=1}^N$ *such that the set* (x, y, p) *is a competitive equilibrium.*

2 The Deterministic Neoclassical Growth Model

2.1 The Baseline Model

The neoclassical growth model consists of a production function taking the time t capital stock k_t and labor services l_t as inputs to produce the final good y_t:

$$y_t = F(k_t, l_t).$$

In addition, each consumer is faced with a budget constraint over dividing current-period output between consumption today and investment:

$$c_t + i_t = y_t,$$

where investment obeys the law of motion for the capital stock

$$k_{t+1} = (1 - \delta)k_t + i_t,$$

assuming a depreciation rate of $\delta \in (0, 1)$.

In a competitive equilibrium each household is faced with a utility maximization problem, subject to their own preferences and budget constraints. Under various sets of assumptions, this problem can be simplified from a n-household economy to one of a representative agent making decisions over the aggregate quantities (or per capita quantities). This representative household then chooses an allocation to solve the following problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \sum_{t=0}^{\infty} p_t[c_t + k_{t+1} - (1 - \delta)k_t] \leq \sum_{t=0}^{\infty} p_t[r_t k_t + w_t l_t]$$

$$0 \leq l_t \leq 1, c_t \geq 0, k_{t+1} \geq 0, k_0 \text{ known.}$$

Here we assume that the agent's discount factor $\beta \in (0, 1)$. As can be seen in the budget constraint, the investment decision over capital has been substituted in. Instead of a social planner choosing how to divide aggregate quantities of output between quantities of consumption and investment, the representative agent chooses consumption and next-period capital, given their income from renting out their capital and labor services. Further, we assume that consumption and capital must be non-negative quantities each period and that the initial capital stock k_0 is a known, fixed quantity.

The above problem is formulated in terms of an Arrow-Debreu competitive equilibrium in which the representative agent makes decisions at date 0, given perfect information about the future and known, fixed prices. Thus, the price p_t represents the price of 1 unit of consumption in period t, measured in period 0 units. On the RHS of the budget constraint, r_t and w_t represent prices paid to the agent for his capital rent and labor services, respectively.

These values have a contamporaneous interpretation. For instance, r_t states that the consumer will receive r_t units of the consumption good c_t for renting out his capital k_t in that period. In combination with the period t price level, $p_t r_t$ states the value of r_t units of the period t consumption good, measured in date 0 units.

The economy is also characterized by a firm, which chooses capital and labor inputs to solve the profit-maximization problem:

$$\max \sum_{t=0}^{\infty} p_t[y_t - r_t k_t - w_t l_t]$$
$$s.t. \quad y_t = F(k_t, l_t) \ \ \forall t$$

and we have the resource feasibility constraint:

$$y_t = c_t + k_{t+1} - (1-\delta)k_t \ \ \forall t.$$

Definition 2.1. *In this setting, a competitive equilibrium is defined as a household allocation decision* $Z^H = \{(c_t, k_{t+1}, l_t, y_t)\}_{t=0}^{\infty}$*, a firm allocation decision* $Z^F = \{(k_t^f, l_t^f)\}_{t=0}^{\infty}$ *and a price system* $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$ *such that, given prices,*

1. *the representative agent maximizes utility subject to the budget constraint, resource constraints and the transversality condition*[4]*,*

2. *the firm maximizes profits subject to its resource feasibility constraint, and*

3. *the aggregate resource feasiblity constraint is met (i.e. markets clear):*[5]

$$F(k_t, l_t) = c_t + k_{t+1} - (1-\delta)k_t \tag{Goods}$$
$$k_t^f = k_t \tag{Capital}$$
$$l_t^f = l_t. \tag{Labor/Leisure}$$

Let's now explicitly examine the conditions that must be met for us to have properly defined a competitive equilibrium. Firstly, notice that leisure (the excess of labor) does not enter the utility function. In this case, we say that the agent inelastically supplies her labor. So, while agents are given a choice of how much time to commit to labor and leisure in this setup, they choose $l^* = 1$ in equilibrium, given that the production function and utility are strictly increasing functions. Thus, the representative agent chooses consumption and capital to maximize utility and the firm chooses levels of capital and employment to maximize profits. Given a Lagrangian construction (with a corresponding λ multiplier on the budget constraint), the representative agent first order conditions are

$$[c_t]: \quad \beta^t u'(c_t) - \lambda p_t = 0 \tag{1}$$
$$[k_{t+1}]: \quad -\lambda p_t + \lambda p_{t+1}[r_{t+1} - (1-\delta)] = 0 \tag{2}$$

[4]The transversality condition can loosely be stated as a boundary condition for models of infinite time. It characterizes the marginal value of capital in far-ahead time periods in such a way that the consumer does not over-accumulate capital in the limit. For more information and detail, refer to the appendix.

[5]Technical note: Specify $\sum_{t=0}^{\infty} p_t = \lim_{T\to\infty} \sum_{t=0}^{T} p_t$. For this problem to be well-defined, we must have that $\sum_{t=0}^{\infty} p_t < \infty$.

and the firm first-order conditions are

$$[k_t^f] : F_k(k_t, 1) = r_t \tag{3}$$

$$[l_t^f] : F_l(k_t, 1) = w_t. \tag{4}$$

Equations (1) and (2) can be joined to yield the Euler Equation

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta[r_{t+1} + 1 - \delta] = \beta[F_k(k_{t+1}, 1) + 1 - \delta]$$

where the last equality comes from exploiting the firm first-order condition on capital in equation (3). This Euler Equation relates the inter-temporal tradeoff that takes place in marginal utility with respect to consumption. When these conditions are met along with those required by resource feasibility, we have a competitive equilibrium.

Now, given that we have a competitive equilibrium, the first and second welfare theorems imply that we may recast this problem as one of a social planner's problem. This allows some simplification to the model: mainly, that we may temporarily forget about prices and embed the production technology in the household budget constraint. Thus, we may solve the problem:

$$v^*(k_0) = \max_{Z^H} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \;\; c_t + k_{t+1} - (1-\delta)k_t = F(k_t, l_t)$$

where we assume the same non-negativity constraints apply and k_0 is given. This representation of the problem is often called the *sequence problem* (SP) as the value function v^* maximizes over an infinite sequence of periods. Greater convenience in solving such a problem may be achieved through trying to solve a recursive problem in the form

$$v(k) = \max_{c,k'} \;\{u(c) + \beta v(k')\}$$

$$s.t. \;\; c + k' - (1-\delta)k = F(k, 1)$$

Note: Often times, we can embed the depreciated capital term inside a newly-defined production function to make everything look neater: $c + k' = \hat{F}(k, 1)$ where $\hat{F}(k, 1) = F(k, 1) + (1-\delta)k$. This maximization is known as the Functional Equation (FE) and it represents the problem as a recursive one, in which an infinite-dimensional problem is reduced to a two-period problem.

Under the assumptions A4.1-A4.2 in SLP, it can be shown that i) the solution to the SP is a solution to the FE for any level of k_0, and ii) the sequence of capital investment decisions that achieves the maximization to the SP is the same as that which achieves the max in the FE, and vice versa (given an added limit condition). These results are embodied in Theorems 4.2-4.5 of the the text.

What is the point of this? Much more is known about the FE in terms of methods to solve and characterize its solution. For instance, we would like to know if the FE is a well-defined problem. To answer such a question, we may notice that the functional equation is a contraction. Define the operator $T : C(X) \to C(X)$ which maps from the space of continuous, bounded functions into itself. Then we may view the FE as

$$Tv(k) = \max_{k'} \;\{u(F(k, 1) + (1-\delta)k - k') + \beta v(k')\},$$

where we have once again subbed the budget constraint into the objective function input. Under Blackwell's conditions, given that this operator i) maps from $C(X) \rightarrow C(X)$, ii) satisfies monotonicity and iii) satisfies discounting given some $\beta \in (0,1)$, it is a contraction with modulus β. Further, under the Contraction Mapping Theorem, we are guaranteed that a fixed point $T\hat{v} = \hat{v}$ exists and it is unique.[6]

Once we have defined the conditions necessary to achieve a unique solution to the maximization problem, we may examine properties of the value function as well as the *argmax* that achieves the maximizing sequence. In particular, we denote the functions that determine the optimal path of consumption and capital to be the *policy functions* $c_t = c(k_t)$ and $k_{t+1} = g(k_t)$, respectively. Notice that any date t consumption or capital level is determined by simply re-iterating the function back to date 0 capital stock k_0.[7]

In addition, we may examine some of the *inheritance properties* that additional assumptions will pass on to the value function v. In particular, we have

i. If u is increasing and F is increasing $\Rightarrow v$ is increasing,

ii. If u is strictly concave and F is concave $\Rightarrow v$ is strictly concave,

iii. If, in addition to the conditions for (ii), u is continuously differentiable and F is differentiable $\Rightarrow v$ is differentiable.

Optimizing with respect to the sole choice variable k', we have the first-order condition:

$$u'(c) = \beta v'(k') \tag{5}$$

and the Envelope condition

$$v'(k) = u'(c)[F_k(k,1) + 1 - \delta] \tag{6}$$

which in combining yields

$$u'(c) = \beta u'(c')[F_k(k') + 1 - \delta],$$

which is simply the Euler Equation that we also achieved in the competitive equilibrium setup.

[6] This theorem involves defining a metric space. Thus, since we are dealing with the space of continuous and bounded functions, we will choose the *sup norm* as our metric where $||f(x) - g(x)|| = \sup_x |f(x) - g(x)|$.

[7] Notice that the period $t+1$ level of capital $k_{t+1} = g(k_t)$ can be rewritten as $k_{t+1} = g(g(k_{t-1}))$ and so forth. Using some abuse of notation, we may write $k_{t+1} = g^{t+1}(k_0)$. Further, the consumption policy function depends on the current-period capital level; thus, $c_t = c(k_t)$ can be rewritten as $c_t = c(g^t(k_0))$.

2.2 Steady States and Convergence to Steady States

Let us consider the economy at a stationary point for the state variable capital. In the context of our neoclassical model, a stationary point is defined by a level of capital k^* such that $g(k^*) = k^*$, which also implies the same, constant level of period consumption. At this point, our Euler equation above simply becomes

$$\frac{1}{\beta} = F_k(k^*) + 1 - \delta. \tag{7}$$

Thus, equation (7) describes a relationship that must be met in order for the economy to be at a point of rest (i.e. a stationary point). While it is useful to analyze the economy at a stationary point, it is of greater interest to discover i) if the point is unique, ii) if such a point even exists and iii) what are the dynamics that determine convergence (or lack thereof) to the stationary point.

In this subsection, we will review a very particular example of a SP/FE problem that does lead to existence and uniqueness with monotone convergence. In general though, such properties of stability should not be taken for granted and solutions may not exist under fairly similar conditions/assumptions imposed on the economy.

Let's now clearly define assumptions on the primitives of the model. We assume that F and u are continuous, strictly increasing, strictly concave and continuously differentiable functions that satisfy the Inada conditions. From these assumptions, it can be shown that there exists a level of capital $\bar{k} > 0$ such that $F(\bar{k}) = \bar{k}$ and this is the maximum attainable capital stock, implying that we can restrict our attention to the interval $[0, \bar{k}]$. Further, F and u are bounded, as well.

Given these assumptions, and using the theorems from SLP Chater 4, we know that there exists a functional equation solution v that is unique, bounded, continuous, strictly increasing and strictly concave. Further, the corresponding policy function for capital $g(k)$ is a continuous function that is strictly increasing in such a way that the solution to v has a unique value for all levels of capital k.

Proposition 2.1. *Given the above model assumptions, the continuous policy function for capital $g(k)$ is strictly increasing in capital.*

Proof. The Envelope condition provides us

$$u'[F(k) - g(k)] = \beta v'[g(k)]$$

where u, F and v are strictly concave functions. Now, suppose instead that g is decreasing. Then, we observe

$$\begin{aligned} k' > k &\Rightarrow g(k') \leq g(k) \\ &\Rightarrow \beta v'[g(k')] \geq \beta v'[g(k)] && \text{(strict concavity of } v\text{)} \\ &\Rightarrow u'[F(k') - g(k')] \geq u'[F(k) - g(k)]. && \text{(Envelope condition)} \end{aligned}$$

By the strict concavity of u, this implies

$$F(k') - g(k') \leq F(k) - g(k)$$

and after re-arranging

$$F(k') - F(k) \leq g(k') - g(k).$$

Because F is strictly increasing and $k' > k$, the LHS of the inequality is positive-valued, whereas the RHS is negative-valued, by assumption. Thus, a contradiction is attained and the policy function g must be strictly increasing. □

What of the stationary points for this model? There exists a trivial stationary point when the incoming capital stock is 0 (i.e. $k = 0$). This is stationary because no capital can be produced tomorrow if there exists no capital today. This is an uninteresting point, and we are more interested in finding interior stationary points in the interval $(0, \bar{k}]$. As we've shown, a stationary point is defined by $\frac{1}{\beta} = F'(k^*) + 1 - \delta$, which implies $k^* = F'^{-1}\{\frac{1}{\beta} - (1+\delta)\}$. Is this actually stationary? Since v is strictly concave, it is a property of such functions that

$$[v'(k) - v'(g(k))][k - g(k)] \leq 0 \quad \forall k \in [0, \bar{k}]$$

and holds with equality (i.e. equal to zero) iff $g(k) = k$. For the first bracketed term, sub in equation for (5) for $v'(k')$ (which is $v'(g(k))$ and sub in equation (6) for $v'(k)$. This leads to

$$[F_k(g(k)) + 1 - \delta - \frac{1}{\beta}][k - g(k)] \leq 0$$

We know that at k^*, the left term is equal to zero, which implies that $k^* = g(k^*)$ and this is in fact a stationary point. What of the dynamics for this system? Since k^* is a unique stationary point in the interior, the inequality above is a strict one when $k \neq k^*$. Thus, when $k < k^*$, concavity of the production function $\Rightarrow F'(k) > \frac{1}{\beta} - (1+\delta)$, and vice versa for capital above the steady state level.

Proposition 2.2. *Let F, u and β satisfy the assumptions above. The corresponding policy function g has two stationary points at $k = 0$ and $k^* = F'^{-1}\{\frac{1}{\beta} - (1+\delta)\}$ and for any initial stock of capital $k_0 \in (0, \bar{k}]$, the sequence $\{k_t\}_{t=0}^{\infty}$, defined by $k_{t+1} = g(k_t)$, converges monotonically to k^*.*

Image of Figure 6.1 on SLP page 135.

2.3 Linear Approximations

There are two relatively simple general methods for deciding whether a particular system is stable. The first is known as the method of Liapounov, which is used for establishing global stability. The second approach is based on linear approximations to the Euler equations. We will examine the implementation and usefulness of this latter approach. In particular, we often care about the policy function g (where g may be a vector of equations) and whether or not it is converging to some constant level of capital and at what rate. If the policy function is a linear one, we can solve the linear system of equations; if the function is nonlinear, we can use counterpart theorems and linear approximations to solve the system. First, let us review some of the linear theory. Much of what follows is taken straight from SLP.

Linear Systems

A system of linear difference equations may be written as

$$x_{t+1} = a_0 + Ax_t \quad \forall t = 0, 1, 2, ... \tag{1}$$

where $x_t \in \mathbb{R}^n$, $a_0 \in \mathbb{R}^n$ and A is a square matrix. Both a_0 and A contain constant scalars. If a point $\bar{x}$ in this sequence were stationary, we'd observe

$$\bar{x} = a_0 + A\bar{x} \Rightarrow \bar{x} = (I - A)^{-1}a_0,$$

given that the matrix $(I - A)$ were non-singular. Let us further define the the deviation $z_t = x_t - \bar{x}$ and now assume that both $(I - A)$ is non-singular and the equation $z_{t+1} = Az_t$ holds in describing equation (1). If this were true, then (1) could be equivalently represented by

$$z_t = A^t z_0. \tag{2}$$

Characterizing the solution to (1) and hence the convergence of the sequence depends upon characterizing the sequence $\{A^t\}$. So, what is known about this sequence? For all t, it is a square matrix, implying the decomposition

$$A = B^{-1}\Lambda B, \tag{3}$$

where B is non-singular and Λ is the Jordan matrix where $\Lambda_{ii} = \Lambda_i$ for all $i = 1, 2, ..., k$ and 0 on the off-diagonal. Each diagonal component Λ_i has the distinct characteristic roots λ_i of A on the diagonal, a 1 above each diagonal element and 0 elsewhere. These characteristic roots are solutions to the equation $\det(A - \lambda I) = 0$.

Why use (3)? Define $w_t = Bz_t$ and use (2) and (3) to show

$$\begin{aligned} w_t =& Bz_t \quad \forall t = 0, 1, 2... \\ =& BAz_{t-1} \\ =& \Lambda Bz_{t-1} \\ =& \Lambda w_{t-1} \\ =& \Lambda^t w_0, \end{aligned} \tag{4}$$

where the *powers* of the Jordan matrix are easily computed (see page 145 of SLP). For a more applicable representation of $w_t = \Lambda^t w_0$, plug in (4) to the LHS and RHS, explicitly writing out the deviations form for z to get

$$\begin{aligned} B(x_t - \bar{x}) &= \Lambda^t B(x_0 - \bar{x}) \\ \Rightarrow x_t &= \bar{x} + B^{-1}\Lambda^t B(x_0 - \bar{x}). \end{aligned} \tag{5}$$

Now, two central theorems to take us home:

Theorem 2.1. *Let a_0 be an n-vector and let A be a square matrix in n dimensions. Suppose the matrix $I - A$ is nonsingular and let $\bar{x} = (I - A)^{-1}a_0$. Then, the $\lim_{t\to\infty} x_t = \bar{x}$ for all sequences $\{x_t\}$ satisfying (1) if and only if the characteristic roots of A are all less than one in absolute value.*

That is to say, given the assumptions, if all the eigenvalues of A lie within the unit $+/-$ unit interval, the linear difference system converges in the limit. When not all of the characteristic roots of A are less than one in absolute values, the initial conditions for the sequence $\{x_t\}$ become a factor, and the following theorem specifies the necessary and sufficient conditions for convergence.

Theorem 2.2. *Let a_0 be an n-vector and let A be a square matrix in n dimensions. Suppose the matrix $I - A$ is nonsingular and let $\bar{x} = (I - A)^{-1}a_0$. Let B be nonsingular and Λ the Jordan matrix for A. Suppose the first m diagonal elements of Λ are less than one in absolute value and the last $n - m$ are euqal to or greater than one. Given the system in (1), the $\lim_{t\to\infty} x_t = \bar{x}$ if and only if x_0 satisfies*

$$x_0 = \bar{x} + B^{-1}w_0$$

with $w_{0i} = 0$ for $i = m + 1, m + 2, ..., n$.

Now, when we move from the clean world of linear difference equations to potentially nonlinear ones, we examine the system

$$x_{t+1} = h(x_t) \quad \forall t = 0, 1, 2, ... \tag{6}$$

where we hope to find a linear approximation to the function h at the stationary point $\bar{x}$. If our initial condition x_0 is close to the stationary point, we can have some confidence that we have a good approximation of the actual function h. I'll now restate the previous two theorems in terms of a linear approximations to a nonlinear system.

Theorem 2.3. *Let $\bar{x}$ be a stationary point of (6) and suppose h is continuously differentiable in a neighborhood N of $\bar{x}$. Let A be the Jacobian matrix of h, evaluated at $\bar{x}$. Further assume that $I - A$ is nonsingular. If the n characteristic roots of A are less than one in absolute value, there exists a neighborhood $U \subseteq N$ such that if $\{x_t\}$ is a solution to (6) with $x_0 \in U$, then $\lim_{t\to\infty} x_t = \bar{x}$.*

What has changed? we are looking at the Jacobian matrix characteristic roots as compared to the roots of the explicitly-given square matrix A in the linear case.

Theorem 2.4. *Let the previous theorem hold but assume that A has m roots less than one in absolute value and $n-m$ that are equal to or greater than one. Then there is a neighborhood $U \subseteq N$ and a continuously differentiable function $\phi : U \to \mathbb{R}^{n-m}$ for which the matrix $[\phi^i_j(\bar{x})]$ has rank $n-m$, such that if $\{x_t\}$ is a solution to (6) with $x_0 \in U$ and $\phi(x_0) = 0$, then $\lim_{t\to\infty} x_t = \bar{x}$.*

Euler Equations

For this part, we will first address the problem in general notation and then lastly apply the results to our neoclassical growth model. Further, As in the previous subsection, we will examine the linear case first, followed by the nonlinear case, using the above theorems. In the neoclassical setting, we care about the evolution and convergence of the capital stock to some stationary level. Thus, we are interested in the policy function g. While this is the ultimate objective, we may not be working with enough information to do so; for instance, we may not know if the function is differentiable or not. Thus, we will instead look at and apply the above results to the Euler equations of the system.

In generalized notation, consider the problem of

$$\max \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

where x_t is a vector of n dimensions. Note: F is <u>not</u> the same as the production function in the neoclassical growth model. Sorry for the confusion, but I am trying to stay consistent with SLP and Chari's lecture. F is commonly referred to as the *return function*. The first input of the return function is the x input while the latter is the y input. In this setting, the Euler equations are

$$0 = F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2}) \tag{7}$$

for all time periods. Given the standard assumptions on boundedness, continuity, increasing and concavity, the optimal solution to this system is unique. For the linear case, let us suppose that F is quadratic, which implies that the first derivatives are linear. The Euler equations can thus be written as

$$0 = F_y + \beta F_x + F'_{xy} x_t + (F_{yy} + \beta F_{xx}) x_{t+1} + \beta F_{xy} x_{t+2}, \tag{8}$$

where the first-derivative vectors are l-dimensional and the second-derivative ones are $l \times l$. Take equation (8) and plug in a stationary point $\bar{x}$ to get

$$\bar{x} = -(F'_{xy} + F_{yy} + \beta F_{xx} + \beta F_{xy})^{-1}(F_y + \beta F_x), \tag{9}$$

assuming that the first term on the RHS is nonsingular. Given (9) and using the same deviation formula $z_t = x_t - \bar{x}$, we can perform painful algebra to get the rewritten Euler equation

$$0 = \beta^{-1} F_{xy}^{-1} F'_{xy} z_t + \beta^{-1} F_{xy}^{-1}(F_{yy} + \beta F_{xx}) z_{t+1} + z_{t+2} \tag{10}$$

which is a second-order difference equation. We can more compactly summarize this within a matrix setup as

$$
\begin{aligned}
Z_{t+1} =& AZ_t \qquad (11)\\
\Rightarrow \begin{bmatrix} z_{t+2} \\ z_{t+1} \end{bmatrix} =& \begin{bmatrix} J & K \\ I & 0 \end{bmatrix} \begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix}
\end{aligned}
$$

where J and K correspond to components of equation (10) above. Further, make note that the A matrix is $2l \times 2l$ in dimension. Now, we make use of the following proposition.

Proposition 2.3. *Assume that F_{xy} and $(F'_{xy} + F_{yy} + \beta F_{xx} + \beta F_{xy})$ are nonsingular, and let the matrix A be as defined in (11). Then, if λ is a characteristic root of A, so is $\frac{1}{\beta\lambda}$.*

Proof. See page 150 of SLP. □

This theorem implies that if we have a characteristic root less than one in absolute value, then there exists a corresponding root that is necessarily greater than one in absolute value, given that $\beta \in (0, 1)$. Further, the A matrix contains $2l$ roots, implying that l are greater than or equal to $\frac{1}{\sqrt{\beta}}$ (in absolute value) while the other l are less. This implies that <u>no more than l roots</u> are smaller than one. The following theorem states that if exactly l roots are smaller than one, we have global stability.

Theorem 2.5. *Let $F : \mathbb{R}^{2l} \to \mathbb{R}$ be strictly concave and a quadratic function. Given that F_{xy} and $(F'_{xy} + F_{yy} + \beta F_{xx} + \beta F_{xy})$ are nonsingular and $\bar{x}$ is the unique stationary point, if the matrix A has l characteristic roots less than one in absolute value, then for every $x_0 \in \mathbb{R}^l$, there exists a unique solution $\{x_t\}$ to the sequence problem such that $\lim_{t\to\infty} x_t = \bar{x}$.*

And now the nonlinear analogue:

Theorem 2.6. *Assume that F is twice continuously differentiable in a neighborhood N around the steady state $(\bar{x}, \bar{x})$. Let F_{xx}, F_{yy} and F_{xy} be the derivative matrices, evaluated at the stationary point and assume the nonsingularity of F_{xy} and $(F'_{xy} + F_{yy} + \beta F_{xx} + \beta F_{xy})$. If the matrix A in (11) has l characteristic roots less than one in absolute value, then for any x_0 sufficiently close to $\bar{x}$, the unique solution $\{x_t\}$ to the sequence problem satisfied $\lim_{t\to\infty} x_t = \bar{x}$.*

The proof defines the nonlinear system with the difference equation

$$
\begin{aligned}
Z_{t+1} =& H(Z_t)\\
\Rightarrow \begin{bmatrix} x_{t+2} \\ x_{t+1} \end{bmatrix} =& \begin{bmatrix} h(x_{t+1}, x_t) \\ x_{t+1} \end{bmatrix}
\end{aligned}
$$

where the Jacobian of the H matrix, evaluated at the steady state, leads to the matrix A from equation (11); that is,

$$
\frac{\partial H(\bar{x}, \bar{x})}{\partial x_{t+1}} = \begin{bmatrix} J & K \\ I & 0 \end{bmatrix} = A.
$$

For the neoclassical growth model, with policy function g, the theorem implies that g must satisfy $\phi[g(k), k] = 0$ for all $k \in U$. Further, if continuously differentiable, we have the derivative $g'(\bar{k}) = -\phi_1^{-1}(\bar{k}, \bar{k})\phi_2(\bar{k}, \bar{k})$, where $\bar{k}$ is the stationary level of capital, through an application of the Implicit Function Theorem.

The Application

Now, let us formally take a look at the deterministic neoclassical growth model. Here, the return function is simply the utility function with the budget constraint subbed in:

$$F(x,y) = u[F(k_t) - k_{t+1}].$$

Once again, apologies for the notation: F on the LHS is the general return function while F on the RHS is the neoclassical production function. Thus, Euler equation (7) is written

$$0 = -u'[F(k_t) - k_{t+1}] + \beta F'(k_{t+1})u'[F(k_{t+1}) - k_{t+2}]. \tag{12}$$

Take a second-order expansion, centered around the stationary point $\bar{k}$:

$$\begin{aligned} 0 =& - F'u''\hat{z}_t + u''\hat{z}_{t+1} + \beta F''u'\hat{z}_{t+1} + \beta F'F'u''\hat{z}_{t+1} - \beta F'u''\hat{z}_{t+2} \\ \Rightarrow 0 =& - \frac{1}{\beta}u''\hat{z}_t + [(1+\frac{1}{\beta})u'' + \beta F''u']\hat{z}_{t+1} - u''\hat{z}_{t+2}, \qquad (F'(\bar{k}) = \tfrac{1}{\beta}) \end{aligned}$$

with deviation notation $\hat{z}_t = k_t - \bar{k}$ and each function is also evaluated with stationary point inputs. Given the model assumptions, we know that $u' > 0$, $F' > 0$ and $F'' < 0$. Using equations (9) and (10) from the general setting,

$$F_{xy} = -\frac{1}{\beta}u'' \quad \text{and} \quad F_y y + (1+\beta)F_{xy} + \beta F_{xx} = \beta u'F''.$$

Thus, the system in (11) becomes

$$\begin{bmatrix} \hat{z}_{t+2} \\ \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{\beta} + \frac{F''/F'}{u''/u'} & -\frac{1}{\beta} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{z}_{t+1} \\ \hat{z}_t \end{bmatrix}. \tag{13}$$

which leads to the characteristic equation for this “A” matrix:

$$\lambda_2 - [1 + \frac{1}{\beta} + \frac{F''/F'}{u''/u'}]\lambda + \frac{1}{\beta} = 0. \tag{14}$$

After some examination, you should be convinced that there exists two positive roots for this equation; further, one is less than one while the other is greater than $\frac{1}{\beta}$.[8] Without loss of generality, let's make λ_1 the smaller of the roots. Thus, we have

$$0 < \lambda_1 < 1 < \frac{1}{\beta} < \lambda_2.$$

After a use of the square matrix decomposition, Theorem 2.6, we get

$$g'(\bar{k}) = \cdots = \lambda_1$$

which leads to an approximation on capital stock convergence

$$k_{t+1} = \bar{k} + g'(\bar{k})[k_t - \bar{k}] = \bar{k} + \lambda_1[k_t - \bar{k}]. \tag{15}$$

Given $\lambda_1 \in (0,1)$, we are guaranteed stable convergence to the steady state $\bar{k}$.

[8]See graph on page 155 of SLP.

3 Measure Theory and Markov Processes

When the neoclassical growth model is extended from a deterministic setting to one involving stochastic shocks to the economy, *measurability* issues arise, as well as the desire to properly characterize (and be confident in) the convergence of relevant measures of probability. Doing so requires a detour into the topic of measure theory. Further, in providing some structure to the nature of the model's stochastic shocks, it is often useful to characterize the process as a *Markov process* so that we may leverage off the topics well-established results. For the course, the main results and theorems are informally presented and the material does not appear to show up in test-taking settings. With that being said, this material is nonetheless essential to fully understand what's going on underneath the hood, so to say. This section is presented at the most general level possible, covering the most basic and necessary definitions and theorems. The following notes are taken almost exclusively from SLP Chapters 3, 11 and 12, as well as from Sergio Ocampo-Diaz's Math 2015 Summer Camp notes.

3.1 Introduction to Measure Theory

Definition 3.1. *Let S be a set and let Σ be a family of subsets of S. Σ is called a* ***sigma algebra*** *if*

i. $\emptyset, S \in \Sigma$,

ii. $A \in \Sigma \Rightarrow A^C \in \Sigma$, *and*

iii. $A_n \in \Sigma \ \ \forall n = 1, 2, ... \Rightarrow \cup_n A_n \in \Sigma$.

Parts ii) and iii) of the definition are referred to as closed under complement and closed under countable union, respectively. The σ-algebra can be as simple as containing just the empty set and the set Ω or, on the opposite extreme, can be the collection of all possible subsets of Ω.

Definition 3.2. *Let a set A be a collection of subsets of S and define Σ_i be the set of sigma algebras that contain A. Then, $B = \cap_i \Sigma_i$ is the sigma algebra* ***generated by*** *A.*

We call the pair (S, Σ) where S is any set and Σ is a sigma algebra is called a **measurable space** and any set $A \in \Sigma$ is called Σ-measurable. The set A is measurable with respect to the sigma algebra Σ if its elements are identifiable in the sense that outcomes represented in A can be told apart from other outcomes, given the information in Σ. A sigma algebra of special importance is the *Borel* sigma algebra.

Definition 3.3. *Let $S = \mathbb{R}$ and Σ the set of open and half open intervals. The* ***Borel algebra****, noted by $\mathbb{B}$ is the sigma algebra generated by Σ. Any set $B \in \mathbb{B}$ is called a Borel set.*

So, the Borel algebra is the smallest sigma algebra that contains all of the open sets in the real line. This notion can of course be generalized to higher, arbitrary dimensions; further, the Borel algebra can be defined for any metric space as the smallest sigma algebra containing all the *open balls*.

Now, let us explicitly define what is meant by *measure*. Given a measurable space (S, Σ), a measure is simply a function $\mu : \Sigma \to \mathbb{R}$ with certain restrictions to guarantee consistency.

Definition 3.4. *Let (S, Σ) be a measurable space. A **measure** is an (extended) real-valued function that satisfies the following conditions:*

i. $\mu(\emptyset) = 0,$

ii. $\mu(A) \geq 0, \quad \forall A \in \Sigma,$

iii. *μ is countably additive; that is, for every disjoint, countable sequence $\{A_n\}_{n=1}^{\infty}$ in Σ,*

$$\mu(\cup A_n) = \sum_{i \in I} \mu(A_n).$$

If $\mu(S) < \infty$, then μ is called a finite measure. Further, if $\mu(S) = 1$ then μ is said to be a probability measure. A triple (S, Σ, μ) where S is a set, Σ is the sigma algebra and μ is the measure is called a **measure space**.[9] The triple is called a **probability space** if μ is a probability measure.

Definition 3.5. *Let (S, Σ, μ) be a measure space. A proposition is said to hold **almost everywhere** or **almost surely** if there exists a set $A \in \Sigma$ such that $\mu(A) = 0$ and the proposition holds only in A^c.*

An example can be drawn from when comparing different functions that are similar to each other. We can say that two functions (f and g) are equivalent almost everywhere, implying that $f(x) = g(x)$ and the $A = \{x : f(x) \neq f(y)\}$ has a corresponding measure $\mu(A) = 0$.

Measurable Functions and Lebesgue Integration

One can think of a function as mapping certain events in a given measure space to outcomes in another measure space. A function is measurable if the sets that induce a given outcome are measurable.

Definition 3.6. *Given a measure space defined by (Ω, Σ, μ), a real-valued function $f : X \to \mathbb{R}$ is measurable with respect to Σ (i.e. Σ-measurable) if*

$$\{x \in \Sigma : f(x) \leq a\} \in \Sigma \quad \forall a \in \mathbb{R}.$$

This definition is not as general as it could be, but it sufficient, given our focus in the course. If the domain in question is a probability space, then f is called a random variable, where the function maps into the Borel sigma algebra. We now introduce two types of functions: an *indicator* and a *simple* function. Let (S, Σ) be a measurable space and consider the indicator function $\chi_A : S \to \mathbb{R}$ defined as

$$\chi_A = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

[9]Often times, it is easier to establish a measure on what is known simply as an *algebra*, which is in some sense smaller than the sigma algebra. The two definitions differ in the sense that algebras pertain to subsets closed under finite unions and intersections, while sigma algebras deal with countable unions and intersections. Generally, it is easier to define a measure on an algebra but to instead work with a sigma algebra and all of its properties. This can be achieved through the application of *extension theorems* which prove the unique equivalence of measures between algebras and sigma algebras. For convenience, these are left out. More detailed notes exist in Chapter 7 of SLP.

for some set $A \in \Sigma$. These functions are measurable, given that the set A is contained within the sigma algebra. Now, consider functions that are finite-weighted sums of indicator functions. A *simple function* is defined as

$$\phi(s) = \sum_{i=1}^{n} a_i \chi_{A_i}(s),$$

where $\{A_i\}_{i=1}^n$ is a sequence of subsets of S and the a_i are scalars. Using this, we can find a convenient result, showing that any measurable function can in fact be expressed as the pointwise limit of a sequence of measurable simple functions.

Theorem 3.1. *Let (S, Σ) be a measurable space. If $f : S \to \mathbb{R}$ is measurable, then there is a sequence of measurable simple functions $\{\phi_n\}$ that converges pointwise to f. If $f \geq 0$, the sequence can be chosen such that*

$$0 \leq \phi_n \leq \phi_{n+1} \leq f \quad \forall n.$$

We now proceed to address the issue of integration and make use of the Lebesgue integral, which is in some sense a generalization of the Riemann integral. The Lebesgue is used because it is applicable to the class of measurable functions, which is in fact larger than the space of Riemann integrable functions. The Lebesgue integral of a function $f : S \to \mathbb{R}_+$ is constructed by taking grids over the range of the function (i.e. the a_i's with $0 \leq a_1 \leq a_2 \leq ... \leq a_n$). From there, we define the set $A_i = \{s \in S : a_i \leq f(s) < a_{i+1}\}$ and the sum $\sum a_i \mu(A_i)$. The integral is simply the limit of this sum as the difference in distance between the y's shrinks to zero. First, look at the integral for simple functions.

Definition 3.7. *Let (S, Σ, μ) be a measure space and let $f : S \to \mathbb{R}_+$ be a simple, measurable function with finitely many values. The Lebesgue integral over the set $A \subseteq S$ is defined as*

$$\int_A f(s) d\mu = \sum_i y_i \mu(A_i)$$

where we define the set A_i as

$$A_i = \{s \in A : f(s) = y_i\}.$$

Now, let's proceed to the more general case of any nonnegative and measurable function.

Definition 3.8. *Let (S, Σ, μ) be a measure space. A measurable function $f : S \to \mathbb{R}_+$ is said to be integrable on a set A if there exists a sequence $\{f_n\}$ of integrable, simple functions converging uniformly to f on A. The Lebesgue integral is thus defined as*

$$\int_A f(s) d\mu = lim \int_A f_n(s) d\mu.$$

An alternative definition makes use of the supremum of a set of simple functions, related to the function f. Under this approach, we would define the integral as

$$\int_A f(s) d\mu = sup\{\int_A f_n d\mu : 0 \leq f_n(x) \leq f(x), \text{where } f_n \text{ are simple functions}\}.$$

3.2 Markov Processes

When dealing with a stochastic environment, measurability issues may arise and this forces us to provide some structure to the environment. For instance, using the functional equation in the presence of stochastic shocks, labeled as s, we may consider the equation

$$v(x,s) = \max_{y \in \Gamma(z,s)} [F(x,y,s) + \beta \int_S v(y,s')\mu(ds')] \tag{1}$$

where x represents the state variable, y the decision variable, chosen from the feasible correspondence Γ. In this problem, the agent observes a stochastic shock today s and must choose y, given the current-period return function F and the discounted, expected value of future utility, represented by the second component of the RHS. This is an *expected* value because the agent doesn't observe tomorrow's shock s' but does observe the disribution of s', according to the measure μ. While reasonable, this approach is limited. In particular, future shocks may change or exhibit serial dependence, while (1) leaves the distribution unchanged. To accommodate this more general setting, we must consider functional equations of the form

$$v(x,s) = \max_{y \in \Gamma(x,s)} [F(x,y,s) + \beta \int_S v(y,s')Q(s,ds')] \tag{2}$$

where $Q(s,\cdot)$ represents a probability measure, depending upon the value of today's shock s. Under the right restrictions, the function Q is called a transition function. We now formally define this function and state some of its properties.

Definition 3.9. *Let (S,Σ) be a measurable space. A **transition function** is a function $Q : S \times \Sigma \to [0,1]$ such that*

i. for each $s \in S$, $Q(s,\cdot)$ is a probability measure on (S,Σ), and

ii. for each $A \in \Sigma$, $Q(\cdot,A)$ is a measurable function, with respect to the sigma algebra.

Part (i) is simple enough to understand: you give me a realized value s from the possible set S and Q maps to the probability of observing any possible value tomorrow s'. Part (ii) is less intuitive and less important for our purposes. Note: the second input to Q can be a set. If this were the case, then Q would provide the probability of any event $s' \in A$ of occurring, given the realization of s. Now, we proceed to define two operators associated with the transition function. Let Q be a transition function on a measurable space (S,Σ). For any measurable function f, define the **Markov operator** Tf as

$$(Tf)(s) = \int f(s')Q(s,ds') \quad \forall s \in S. \tag{3}$$

Recall that $Q(s,\cdot)$ is a probability measure; thus, Tf is well defined and we can interpret $(Tf)(s)$ as the expected value of f next period, given today's realization of s. Next, for any probability measure μ on (S,Σ), define the **adjoint of T** $T^*\mu$ as

$$(T^*\mu)(A) = \int Q(s,A)\mu(ds) \tag{4}$$

for all $A \in \Sigma$. Given that $Q(\cdot,A)$ is a measurable and bounded function (by definition), the function $T^*\mu$ is well defined, as well. Further, we may think of $(T^*\mu)(A)$ as the probability that the state next period lies in the set A, given the current realization of s, today.

A few notational inclusions follow. For any measurable space (S, Σ), $M^+(S, \Sigma)$ is the space of nonnegative, measurable, extended real-valued functions, $B(S, \Sigma)$ is the space of bounded, measurable, real-valued functions and $\Lambda(S, \Sigma)$ is the space of probability measures on the aforementioned measurable space. Given this, we now post a couple results with respect to the operators Tf and $T^*\mu$.

Theorem 3.2. *The operator Tf defined in (3) maps $Tf : M^+(S, \Sigma) \to M^+(S, \Sigma)$.*[10]

Theorem 3.3. *The operator $T^*\mu$ in (4) maps $T^*\mu : \Lambda(S, \Sigma) \to \Lambda(S, \Sigma)$.*

Now, we proceed with an important theorem that establishes the connection between the two operators.

Theorem 3.4. *Let Q be a transition function on the measurable space (S, Σ) and consider the Markov operator and the adjoint of T. For any function $f \in M^+(S, \Sigma)$,*

$$\int (Tf)(s)\mu(ds) = \int f(s')(T^*\mu)(ds') = \int \int f(s')Q(s, ds')\mu(ds) \quad \forall \mu \in \Lambda(S, \Sigma). \tag{5}$$

Equation (5) computed the expected value of a function tomorrow, and the theorem shows that one can compute this expected value with either operator. This is particularly useful in a dynamic programming when we are interested in computing the expected *continuation utility*, which is composed of the value function as in equation (2). The transition function can be used not just for period-to-period probabilities but for n-step probabilities. In particular,

$$Q^{n+1}(s, A) = (TQ^n(\cdot, A))(s) = \int Q^n(s', A)Q(s, ds') \tag{6}$$

is the probability of ending up in state A $n+1$ periods from now, conditioned on the realization of shock s today. This can be computed through successive iterations of the Markov operator or through integrating over the transition function raised to the n^{th} power.

Sometimes we may wish to impose stronger properties on the stochastic environment (specifically, on the T operator) to attain different results. Two mentioned here are the Feller property and the notion of monotonicity of the transition function.

Definition 3.10. *A transition function Q on (S, Σ) has the* ***Feller property*** *if the associated operator T maps the space of bounded continuous functions on S into itself; that is, $T : C(S) \to C(S)$.*

This is a stronger assumption as the space of continuous bounded functions is obviously a subset of $B(S, \Sigma)$.

Definition 3.11. *A transition function Q on (S, Σ) is* ***monotone*** *if the associated operator T has the property that for every nondecreasing function $f : S \to \mathbb{R}$, the function Tf is also nondecreasing.*

For example, if we have some return (or value) function that is increasing in its state variable, the expected value of that function (as represented by the Markov operator Tf) will also be increasing in the state variable.

Let's introduce some notation to examine (partial) histories of stochastic shocks. We use $s^t = (s_1, ..., s_t)$ to represent a particular history of shocks, up to period $t < \infty$. In addition, let (S, Σ) be a measurable space and denote

$$(S^t, \Sigma^t) = (S \times ... \times S, \Sigma \times ... \times \Sigma)$$

[10] Theorem 4.2 also implies that the operator Tf also maps $Tf : B(S, \Sigma) \to B(S, \Sigma)$.

as the product space, where each S, and Σ is repeated t times above. We can then define the probability measure $\mu^t(s_0, \cdot)$ as

$$\mu^t(s_0, B) = \int_{A_1} ... \int_{A_{t-1}} \int_{A_t} Q(s_{t-1}, ds_t)Q(s_{t-2}, ds_{t-1})...Q(s_0, ds_1)$$

for any rectangle $B = A_1 \times A_2 \times ... \times A_t \in \Sigma^t$. Notice that this probability measure is not just the probability of ending up in state A_t, given initial shock s_0 but is the probability of arriving at A_t hrough the particular *string* of intermediate sets specified in B.

Definition 3.12. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A* ***stochastic process*** *on $(\Omega, \mathcal{F}, P)$ is an increasing sequence of sigma algebras (known as a filtration) $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}$, the measurable state space (S, Σ) and a sequence of functions $\sigma_t : \Omega \to S$ such that each σ_t is measurable with respect to $\mathcal{F}_t$.*

Each function σ_t is a random variable that takes a value on the set of possible shocks S, given the occurrence of some event on Ω. Actually, a stochastic process is simply the family of random variables σ_t presented above, whereas the definition is technically describing a stochastic process that is **adapted** to the filtration $\{\mathcal{F}\}$. In most cases, we simply consider $(S, \Sigma) = (\mathbb{R}, \mathbb{B})$ where $\mathbb{B}$ represents the Borel sigma algebra. The reason that the sigma algebras are increasing is because each date t sigma algebra $\mathcal{F}_t$ represents the sets of all possible random shocks today, as well as the all possible histories. Therefore, it naturally follows that the number of possible histories grows as t becomes larger. Let $\omega \in \Omega$ be a fixed element of the sigma algebra; then, $(\sigma_1(\omega), \sigma_2(\omega), ...)$ is called the **sample path** of the stochastic process.

Now, let's define the probability of a certain path. In particular, let $C \in S^n$ and write

$$P_{t+1,...,t+n}(C) = P(\{\omega \in \Omega : (\sigma_{t+1}(\omega), ..., \sigma_{t+n}(\omega)) \in C\}) \tag{7}$$

as the probability that the sample path lies within the set C from periods $t + 1$ to $t + n$. A stochastic process is deemed **stationary** if the probabilities $P_{t+1,...,t+n}(C)$ are independent of t, for all n and all $C \in S^n$. Before defining a Markov process, we need to quickly establish the notion of conditional probability. $P_{t+1,...,t+n}(C|a_{t-s}, ..., a_{t-1}, a_t)$ is the conditional probability of the event $\{\omega \in \Omega : [\sigma_{t+1}(\omega), ..., \sigma_{t+n}(\omega)] \in C\}$, given that the event $\{\omega \in \Omega : \sigma_\tau = a_\tau, \tau = t - s, ..., t - 1, t\}$ has occurred. Thus, given additional information about what is known to have occurred in past periods has an affect upon the future probability of events.

Definition 3.13. *A* ***(first-order) Markov process*** *is a stochastic process with the property that*

$$P_{t+1,...,t+n}(C|a_{t-s}, ..., a_{t-1}, a_t) = P_{t+1,...,t+n}(C|a_t) \tag{8}$$

for all $t = 2, 3, ...$, for all $n = 1, 2, ...$, for all $s = 1, 2, ..., t - 1$ and for all $C \in S^n$.

The first-order process simply implies that the probability of some future path is independent of all past observations except for the most recent period. This can of course be generalized to higher-orders. If the conditional probability $P_{t+1}(A|a)$ is independent of t for all $a \in S$, the Markov process is said to have **stationary transitions**, which is different from a stationary process. One more definition!

Definition 3.14. *Let $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ be measurable spaces. A* ***stochastic kernel*** *on $\{X, \mathscr{Y}\}$ is a function $P : X \times \mathscr{Y} \to [0, 1]$ such that* *i. for each $a \in X$, $P(a, \cdot)$ is a probability measure on $(Y, \mathscr{Y})$, and*

ii. for each $B \in \mathscr{Y}$, $P(\cdot, B)$ is an $\mathscr{X}$-measurable function.

Thus, a transition function is a stochastic kernel where the two spaces $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ coincide with one another.

3.3 Convergence of Markov Processes

In the deterministic neoclassical growth model, we were able to characterize some properties of the steady state for the capital stock (denoted $\bar{k}$ or k^*) and examine settings in which certain model assumptions on the primitives led to global, monotone convergence. Within the context of the model, this implied that given any initial stock of capital, rational agents would continually choose future capital k_{t+1} in a way that eventually converged to a constant k^*, and there the model would stay forever. With the introduction of stochastic shocks to the model, the notion of stationary points and convergence must be adjusted since Markov processes do not, in general, converge to a constant value. Recall that for the stochastic growth model, the state of the economy in any given period is summarized by both the aggregate capital stock and the current period shock. Also in this model, we are given/derive the law of motion for capital ϕ, the policy function for capital g and the transition function for the stochastic shock Q. Moving forward, the idea is that if we are given ϕ, g and Q, we can formulate another transition function P to determine the probabilistic evolution of the *states* of the economy through time.

3.3.1 Finite State Space and Markov Chains

Recall that for a transition function Q on a measurable space (S, Σ), the adjoint operator T^* maps the set of probability measures$\Lambda(S, \Sigma)$ into itself. Thus, if we have the date 0 probability measures μ_0, then we can use the adjoint operator to recursively arrive at the date t measure by means of $\mu_t = T^* \mu_{t-1}$. Thus, if we are interested in the convergence of the stochastic process $\{s_t\}$, it makes sense that we should be interested in convergence of the probability measure $\{\mu_t\}$, as well. We call a probability measure μ^* an **invariant probability measure** if being the probability measure over s_t in period t implies it is also the probability measure over s_{t+1} in period $t+1$. Put more technically, an invariant probability measure is a fixed point of the adjoint operator; that is, $T^* \mu^* = \mu^*$. This section deals with defining properties of the transition function Q that are sufficient to guarantee existence, uniqueness, convergence and robustness.

To begin, let us first start with cases in which the state space S consists of a finite number of elements, such that $S = \{s_1, ..., s_l\}$ and Σ contains all the subsets of S. A Markov process on a finite state space is called a **Markov Chain**. When S is a finite set, a probability measure on the space is a vector p in the l-dimensional simplex: $\Delta^l = \{p \in \mathbb{R}^l : p \geq 0 and \sum p_i = 1\}$. Thus, we use the **Markov matrix**: the $l \times l$ matrix Π such that individual elements $\pi_{ij} = Prob(s_i, \{s_j\})$. In this context, individual rows represent todays shock s, whereas the columns represent states tomorrow. Therefore, each row must sum to unity, across all future states of the world. In addition, let e_i be an l-dimensional row vector with 1 in the i^{th} position and zeros elsewhere. Now, a couple quick tricks:

i. If the current state is s_i, then denote todays probability distribution as e_i such that the distribution over next period's state is

$$e_i\Pi = \pi_i,$$

ii. If the unconditional distribution over period t's state is the p, then the distribution over the state in period $t+1$ is given by

$$\hat{p}_j = \sum_{i=1}^{l} p_i \pi_{ij}$$

for all states tomorrow $j = 1, ..., l$.

For (i), because today's state has been realized and is s_i, there is certainty that s_i has occurred; thus, we may use the distribution e_i. For (ii), we do not know the outcome of state t and our instead given a distribution of outcomes. To arrive at a probability of state j in period $t+1$, we multiply the probability of any state i in $t+1$ times the probability of going from state i to j in period $t+1$, and then sum these terms. Note: (ii) is a scalar.

Part (ii) above generalizes for the entire Markov matrix Π. In particular, if the probability distribution over the state in period t is p, then the probability distribution over the state two periods ahead is $p\Pi^2$ and this can be done for arbitrary n-step transitions. Further, if the initial state shock is s_i, the the probability distribution over states n periods ahead is $e_i\Pi^n$. Thus, it makes sense to look at the sequence $\{\Pi^n\}$ which is a sequence of Markov matrices.

Definition 3.15. *A set $E \subseteq S$ is an **ergodic set** if $prob(s_i, E) = 1$ for $s_i \in E$ and no proper subset of E has this property.*

Essentially, when dealing with Markov matrices, an ergodic set is a set of elements in the state space where there is zero probability of leaving that set. This notion is best understood through examples. Refer to Chapter 11 of SLP and/or other internet sources for this aid. When each row vector of the Markov matrix Π converges to a constant for the limit of Π^n, we call this an **invariant distribution**.

Theorem 3.5. *Let $S = \{s_1, ..., s_l\}$ be a finite set and let the Markov matrix Π define transition probabilities on S. Then*

a. S can be partitioned into $M \geq 1$ sets,

b. The sequence $\{\frac{1}{n}\sum_{k=0}^{n-1}\Pi^k\}_{n=1}^{\infty}$ converges to a stochastic matrix Q, and

c. Each row of Q is an invariant distribution.

Note that this theorem does not say that the sequence $\{\Pi^k\}$ converges; instead, it is the average of matrices in the sequence that will converge. While multiple ergodic sets can occur, we would like to find a set of conditions to guarantee that only one unique ergodic set arises.

Theorem 3.6. *Let $S = \{s_1, ..., s_l\}$ be a finite set with the Markov/transition matrix Π on states in S. Then Π has a unique ergodic set if and only if $\exists n \geq 1$ for which at least one column of Π^n has all strictly positive probabilities. In this case, Π has a unique invariant distribution p^* such that each row of Q is equal to p^*. Thus, for any initial distribution p_0, $p_0 Q^* = p^*$.*

The necessary and sufficient requires that there exists some state j such that for all states i, there is a positive probability of ending up in state j in n periods. To prove the unique convergence of the sequence $\{p_0\Pi^k\}$ for any given initial distribution p_0 requires use of the Contraction Mapping Theorem. Lemma 11.3 and Theorem 11.4 of SLP provide the necessary machinery to achieve such a result and they are left out here to save on space. To summarize, the main results are

1. S can always be partitioned into 1 or more ergodic sets and the sequence of averages $\{\frac{1}{n}\sum_{k=0}^{n-1}\Pi^k\}_{n=1}^{\infty}$ converges to a stochastic matrix Q where the rows of Q are invariant distributions.
2. If the conditions of Theorem 3.6 hold, there is a unique ergodic set. In this case, all the rows of Q are identical where each row vector p^* is the same unique invariant distribution, and
3. If the conditions of SLP Theorem 11.4 hold, the sequence $\{\Pi^n\}$ converges to Q and the sequence $\{p_n\} = \{p_0\Pi^n\}$ converges to the invariant distribution p^*.

3.3.2 Weak Convergence

We now proceed to define more general cases of convergence in which the state space is potentially not a finite set as well as provide two separate definitions of convergence.

Definition 3.16. *Let (S,ρ) be a metric space, let $\{\mu_n\}$ and μ be measures in $\Lambda(S,\mathscr{B})$ and let $C(S)$ be the space of bounded, continuous, real-valued functions on S. Then $\{\mu_n\}$* ***converges weakly*** *to μ if*

$$\lim_{n\to\infty}\int f d\mu_n = \int f d\mu \quad \forall f \in C(S).$$

This definition pertains to the limits of the expected values of the class of function in $C(S)$. Weak convergence is very often all we care about in the context of describing the dynamics of an economic system. However, there is a stronger concept that is worth mentioning.

Definition 3.17. *Let (S,Σ) be a measurable space and let $\{\mu_n\}$ and μ be measures in $\Lambda(S,\Sigma)$. Then $\{\mu_n\}$* ***converges strongly*** *to μ if*

$$\lim_{n\to\infty}\int f d\mu_n = \int f d\mu \quad \forall f \in B(S,\Sigma)$$

and if in addition the rate of convergence is uniform for all $f \in B(S,\Sigma)$ such that $||f|| = \sup_{s\in S}|f(s)| \leq 1$

where $B(S,\Sigma)$ refers to the space of bounded, measurable functions with respect to the sigma algebra Σ. SLP prefers to present the topic of proving uniqueness and convergence in the *strong* sense because it parallels the same methods employed for Markov chains in a finite state space. While this is helpful, it is excluded from these notes, as we generally only care about characterizing weak convergence of Markov processes in this course and in the SLP text.

For the remainder of this section, we will be concerned with sequences $\{\mu_n\}_{n=0}^{\infty}$ of probability measures defined by $\mu_{n+1} = T^*\mu_n$ where μ_0 is the initial probability measure and T^* is the adjoint operator. A probability measure μ^* is called invariant under T^* if it is a fixed point of the operator; that is, $T^*\mu^* = \mu^*$. We will assume that (S,ρ) is a metric space and $\mathscr{B}$ is the Borel sigma algebra. The objective is to find one or more families of sets $\mathscr{A} \subset \mathscr{B}$ such that convergence of the measures of sets in $\mathscr{A}$ implies weak convergence. The following theorem provides several different ways in which to characterize weak convergence.

Theorem 3.7. *Let $\{\mu_n\}$ and μ be probability measures on $(S, \mathscr{B})$. Then the following four conditions are equivalent:*

a. $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu \quad \forall f \in C(S)$,

b. for every closed set F, $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$,

c. for every open set G, $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$, *and*

d. $\lim_{n\to\infty} \mu_n(A) = \mu(A)$, *for every set $A \in \mathscr{B}$ with $\mu(\partial A) = 0$*

where ∂A is the boundary of A.

Part (a) corresponds to the previously provided definition of weak convergence whiles parts (b) and (c) provide *set* definitions for convergence. Theorem 3.7 highlights the variety of ways in which one can prove weak convergence, given the problem at hand. Note: The definitions of convergence above are also known as *convergence in distribution*, which may be a more familiar term and concept to you. Theorem 3.8 provide a criterion for weak convergence in a finite-dimensional Euclidean space.

Theorem 3.8. *Let $\{\mu_n\}$ and μ be probability measures on $(S, \mathscr{B})$ where $S \subseteq \mathbb{R}^l$, and let $\mathscr{A} \subseteq \mathscr{B}$ be a family of sets such that*

a. $\mathscr{A}$ is closed under finite intersection, and

b. for every $x \in S$ and $\epsilon > 0$, there exists $A \in \mathscr{A}$ such that $x \in \mathring{A} \subseteq A \subseteq b(x, \epsilon)$,

where $\mathring{A}$ denotes the interior of A relative to S and $b(x, \epsilon)$ is the epsilon ball around x with the metric ρ. Then $\mu_n(A) \to \mu(A)$ for all $A \in \mathscr{A}$ implies $\mu_n \to \mu$.

Now, a brief review of the main theorems that provide sufficient conditions for the weak convergence of a Markov process. In what follows, we assume that the set $S \subseteq \mathbb{R}^l$. We use the transition function P on the measurable space $(S, \mathscr{B})$ with the operators $T : B(S, \mathscr{B}) \to B(S, \mathscr{B})$ and $T^* : \Lambda(S, \mathscr{B}) \to \Lambda(S, \mathscr{B})$. Further, recall that a transition function P has the **Feller property** if for any bounded, continuous function f, the function Tf is also continuous (that is, $T[C(S)] \subseteq C(S)$). Also, a transition function P is **monotone** if for any bounded, increasing function f, the function Tf is also increasing.

Theorem 3.9. *If $S \subset \mathbb{R}^l$ is compact and P has the Feller property, then there exists a probability measure that is invariant under P.*

For the proof, such an invariant measure turns out to be the sample average of the iterations of the adjoint operator.[11] Theorem 3.9 does not rule out the existence of multiple invariant distributions in the limit. To achieve results for convergence to a unique, invariant distribution, we use the additional stronger assumptions of monotonicity and Assumption 3.1

[11] Theorem 12.10 with proof in SLP.

Assumption 3.1. *There exists $c \in S, \epsilon > 0$ and $N \geq 1$ such that $P^N(a, [c, b]) \geq \epsilon$ and $P^N(b, [a, c]) \geq \epsilon$.*

This assumption is referred to as a *mixing condition* used to ensure uniqueness of the invariant measure and further imposes boundedness conditions on the expected value of any bounded, measurable and nondecreasing function.

Theorem 3.10. *Let $S = [a, b] \subset \mathbb{R}^l$. If P is monotone, has the Feller property and satisfies assumption 3.1, then P has a unique invariant probability measure μ^* and $T^*\mu_0 \to \mu^*$ for all $\mu_0 \in \Lambda(S, \mathscr{B})$.*

Thus, Theorem 3.10 characterizes conditions of a transition function that are sufficient for guaranteeing weak convergence to an invariant probability measure. These results have applications within the framework of the course, including topics in stochastic dynamic programming, incomplete markets and more.

4 The Stochastic Neoclassical Growth Model

Convergence to a steady state in the standard growth model implies smooth convergence to some constant level. Modern macroeconomic data seems to contradict this description, as aggregate time series often show continued growth in the levels of consumption, capital, etc. Thus, a standard fix to such issues is adding some trend in growth to the data, in terms of technology and population. Even then though, these time series do not conform well to historic data, as history is wrought with the ups and downs of the business cycle. Thus, a logical extension to the standard model is the introduction of uncertainty or randomness. While this can be accomplished in different ways, our simplified model will simulate randomness through stochastic shocks to the productivity factor associated with the production function. While the solution approach is made more complicated, much of the same methods and results are achieved here as in the deterministic case.

As a first step, let us slightly augment the neoclassical production function to include a productivity factor A_t which is a random variable, susceptible to a finite set of possible values in any given period. Thus, period t output is now

$$A_t(s^t)F(k_t, l_t),$$

where $A_t(s^t)$ is a function of the history of random shocks $s^t = (s_0, s_1, ..., s_t)$. Here, the superscript implies the history while the subscript implies the realized shock at the given date. Further, we assume that the shocks occur in some discrete and finite space. In this sense, all of the quantities in the model are now functions of this randomness; thus, we write $(c_t(s^t), k_{t+1}(s^t), y_t(s^t), l_t(s^t))$.

Under this setting, the representative agent is faced with the following maximization problem:

$$v^*(k_0) = \max_{Z^H} \sum_t \sum_{s^t} \beta^t \mu_t(s^t) u(c_t(s^t))$$

$$s.t. \quad \sum_t \sum_{s^t} p_t(s^t)[c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1})] = \sum_t \sum_{s^t} p_t(s^t)[w_t(s^t)l_t(s^t) + r_t(s^t)k_t(s^{t-1})]$$

along with the usual non-negativity constraints. Note that the measure $\mu_t(s^t)$ is a probability measure such that $\sum_{s^t} \mu_t(s^t) = 1$.[12] In addition, the firm chooses inputs to maximize profits in the problem

$$\max_{Z^F} \sum_t \sum_{s^t} p_t(s^t)[y_t(s^t) - r_t(s^t)k_t^f(s^t) - w_t(s^t)l_t^f(s^t)].$$

$$s.t. \quad y_t(s^t) \leq A(s^t)F(k_t^f(s^t))$$

and the aggregate resource feasibility requirements

$$k_t(s^{t-1}) = k_t^f(s^t)$$
$$l_t(s^t) = l_t^f(s^t)$$
$$c_t(s^t) + k_{t+1}(s^t) = A(s^t)F(k_t^f(s^t)).$$

Notice that the firm's demand for capital inputs, which is influenced by the date t shock, is restricted by the amount of capital that consumers chose in the previous period, which is only influenced by shocks up to date $t-1$.

To add more structure to the nature of the stochastic shocks, let us assume that the distribution of a specific date $t+1$ shock s_{t+1} behaves as a first-order Markov Chain and may be written as

$$\mu(s_{t+1}|s^t) = \mu(s_{t+1}|s_t) \equiv \text{the probability of event } s_{t+1} \text{ given the realized } s_t.$$

In the general setting, a shock is conditioned on the history of shocks s^t, whereas now it simply is a function of last period's value. These probabilities can be further represented by a transition matrix P where the element $P_{i,j}$ represents the probability of s_{t+1} = state j tomorrow, given s_t = state i, today. Thus, rows represent the current state and columns represent possible future states. As covered in Chapter 3, we must have $\sum_j P_{i,j} = 1 \quad \forall i$.

The first welfare theorem implies that we may rewrite the representative agent's problem as the social planner's problem:

$$\max_{\{c_t\}_{t=0}^{\infty}, \{k_{t+1}\}_{t=0}^{\infty}} \sum_t \sum_{s^t} \beta^t \mu(s^t|s_{t-1}) u(c_t(s^t))$$

$$s.t. \; c_t(s^t) + k_{t+1}(s^t) = A(s^t)F(k_t(s^{t-1})) + (1-\delta)k_t(s^{t-1})$$

or we may equivalently rewrite this stochastic sequence problem as the stochastic functional equation, also known as the Bellman equation:

$$v(k,s) = \max_{c,k'} \{u(c) + \beta \sum_{s'|s} \mu(s'|s) v(k',s')\}$$

$$s.t. \quad c + k' \leq A(s)F(k,s) + (1-\delta)k$$

where both problems are subject to the same non-negativity constraints on consumption, labor and capital. Notice what has changed for the FE: in the deterministic neoclassical growth model, the *state* of the economy was entirely summarized by the level of capital k, whereas now, the state of the economy depends upon the capital

[12] The summation over $\sum_{s^t}$ can be interpreted as the possible realizations of s_i that may occur at date t, given the observed history $(s_0, s_1, ..., s_{t-1})$. Thus, the probability measures $\mu_t(\cdot)$ are represented as unconditional probabilities, but technically depend upon the path of shocks that have already occurred.

stock and current-period shock s. Under weak conditions, we have $v^*(k,s) = v(k,s)$ and through applications of the Blackwell conditions, we can once again show that a unique solution exists.[13]

Definition 4.1. *The conditional distribution function $\mu(s'|s)$ is monotone if for all increasing functions $h(s')$,*

$$\sum_{s'} h(s')\mu(s'|s) \geq \sum_{s'} h(s')\mu(s'|\hat{s})$$

if and only if $s \geq \hat{s}$.

Notice that this definition coincides with the more general definition of a monotone transition function: the *Markov* operator, which computes the expected value of the increasing function h, is an increasing function, as well. Let us further define (without loss of generality) a natural ordering of states that exists for the random variable $A(s)$. That is to say, let us assume that $A(s)$ increases in s. Under this stochastic setup, the same inheritance properties can be endowed upon the value function. These are

i. If u and F are increasing functions, this implies v is increasing in k.

ii. If u is strictly concave and F is concave, this implies v is strictly concave.

iii. If, in addition to the conditions for (ii), u and F are continuously differentiable, this implies v is continuously differentiable.

iv. If μ is monotone, then v is increasing in s.

Through the Theorem of the Maximum, the policy functions $c(k,s)$ and $g(k,s)$ are single-valued, continuous functions. Further, by the monotonicity of μ, these policy functions are increasing in both k and s. Thus, higher levels of capital and higher levels of *positive* productivity shocks, leads to an increase in the optimal level of both consumption and capital investment.[14]

5 Search and Unemployment Models

A useful subset of models that lends itself to stochastic dynamic programming is those dealing with search and unemployment for some agent worker. These models attempt to characterize the matching process that takes place between a worker and hiring firms. This section will present a very basic model (known as the McCall model) to introduce some of the fundamental concepts. One thing that becomes apparent very quickly with these models is that there exists a variety of small adjustments that can be introduced into the modeling environment in an attempt to capture real-life dynamics in the job matching market. While these models may be only briefly discussed in recitation, they do seem to make appearances on course exams as well as the midterms; therefore, they merit some attention. The following notes are taken entirely from Chapter 6 of Ljungqvist and Sargent's text.

[13] For the case in which the distribution of the shocks is continuous, the stochastic Bellman equation simply integrates over the domain of possible values. We use the form $v(k,s) = max\{u(c,s) + \beta \int v(k',s')dG(s'|s)\}$, where $G(s'|s)$ is some conditional distribution CDF.

[14] Note that when the stochastic process s is assumed to be independent and identically distributed, the conditional distribution function $\mu_t(s_{t+1}|s_t)$ simply becomes $\mu(s_{t+1})$ and is defined as monotone, trivially.

5.1 Mathematical Preliminaries

First, we must briefly review a couple useful results from probability theory. For this section, we will utilize some cumulative probability distribution function $F(p) = Prob\{P \leq p\}$ where we assume that $F(0) = 0$, implying that p only takes on non-negative values. Lastly, we also assume that these distributions have some upper bound B such that we observe no p larger and where $F(B) = 1$. Together, these imply that there exists zero probability of observing an observation $\tilde{p}$ outside of the interval $[0, B]$. In addition, given some CDF, the expected value of the random variable p, denoted $E[p]$ is defined by

$$E[p] = \int_0^B pdF(p) = \int_0^B pf(p)dp, \tag{1}$$

where $f(p)$ denotes the probability density function for the random variable. Given some manipulation, we also have the equivalent expression for the mean

$$E[p] = B - \int_0^B F(p)dp.^{15} \tag{2}$$

In this section, we will deal with situations in which the economic agent must maximize his decision over a set of discrete choices. Thus, we will encounter use of the *max* operator. Note, for n independent and identical draws of p_i from the cumulative distribution function $F(p)$, the $Prob\{max(P_1, P_2, ..., P_n) < p\} = F(p)^n$.

Lastly, we introduce the concept of a *mean-preserving spread*. As can be grasped from the title, this refers to analyzing/comparing multiple distributions which are characterized by the same mean. In particular, we consider a class of distributions, indexed by some parameter r in the set R. We assume that $F(0, r) = 0$ and $F(B, r) = 1$ for all possible distributions in the set/class. These distributions carry the same expected value for the random variable p; thus, we write

$$\int_0^B [F(p, r_1) - F(p, r_2)]dp = 0. \tag{3}$$

Further, two distributions, indexed by r_1 and r_2 satisfy the *single-crossing property* if there exists a $\hat{p} \in (0, B)$ such that

$$F(p, r_2) - F(p, r_1) \leq 0 \quad \text{when} \quad p \geq \hat{p} \tag{4}$$

and vice versa, whereby both inequality signs are flipped. Essentially, this property states that as p increases in value from 0 to B, the difference will switch from negative to positive value at only one point $\hat{p}$. These are CDFs so this means that at $\hat{p}$, there is equal probability of observing values below (or above) $\hat{p}$ across the two distributions. Further, there is more probability mass to the left of $F(p, r_1)$ for all p larger than $\hat{p}$. When properties (3) and (4)

[15] This can be obtained by applying integration-by-parts formula $\int_a^b udv = uv - \int_a^b vdu$ and evaluate at the endpoints $\{0, B\}$. Thus, we have

$$\begin{aligned}\int_0^B pdF(p) &= pF(p)\Big|_0^B - \int_0^B F(p)dp \\ &= [B(1) - 0F(0)] - \int_0^B F(p)dp \\ &= B - \int_0^B F(p)dp.\end{aligned}$$

are observed together, we say that the distribution indexed by r_2 has been obtained from the distribution indexed by r_1 by a mean-preserving spread. These two conditions, imply

$$\int_0^y [F(p, r_2) - F(p, r_1)]dp \geq 0 \quad \forall y \in [0, B]. \tag{5}$$

5.2 The McCall Model

Here we present McCall's model of intertemporal job search. We consider an unemployed worker who is searching for a job under the following circumstances: each period the worker draws an *i.i.d.* offer from the wage distribution $F(w)$ where $F(\cdot)$ has the same properties as before, including the boundary conditions $[0, B]$. The worker may either reject this wage offer, and receive unemployment compensation $c > 0$ that period, or may accept the offer, receiving w for the period and every period forward. This game takes place over an infinite horizon. This model is indeed simple as neither quitting or firing is permitted.

In period t, the worker's income is denoted y_t. This worker chooses whether to accept or reject the offer in attempt to maximize lifetime expected utility $\sum_{t=0}^{\infty} \beta^t y_t$, where the discount factor β is assumed between 0 and 1. What are the factors that determine acceptance or rejection in this environment? Given that the worker is aware of the properties of the cumulative distribution function, he may choose to forego an offer $w_t > c$ today because it is highly probable that future offers will be even larger than w_t. Thus, he will incur some opportunity cost in the pursuit of a higher future wage. The value function $v(w)$ satisfies the Bellman equation

$$v(w) = \max_{\text{accept,reject}} \{\frac{w}{1-\beta}, c + \beta \int_0^B v(w')dF(w')\}. \tag{6}$$

A common way to analyze these problems is to identify/determine a reservation wage $\bar{w}$; that is, a wage at which the worker is indifferent between accepting or and rejecting the offer $\bar{w}$. Put differently, any offer above the reservation wage, the worker will accept, and any offer below the reservation wage, the worker will reject and continue to search. Given this, we can reformulate the Bellman equation to

$$v(w) = \begin{cases} \frac{\bar{w}}{1-\beta} = c + \beta \int_0^B v(w')dF(w'), & \text{if } w \leq \bar{w} \\ \frac{w}{1-\beta}, & \text{if } w \geq \bar{w}, \end{cases} \tag{7}$$

where the decision to reject is on top and the decision to accept on bottom. Since this takes place over an infinite horizon, accepting the offer w today is accepting the infinite discounted stream of w, leading to $\frac{w}{1-\beta}$. Notice that the decision to reject is equal to the infinite discounted sequence of the reservation wage. This must hold, by definition of the reservation wage. Thus, given the top row of the value function, the reservation wage is $\bar{w} = (1-\beta)[c + \beta \int_0^B v(w')dF(w')]$.

To gain some more insight into this representation of the reservation wage, we can look at a couple other ways to see how the particular value of the wage is determined. Specifically, evaluate the value function at $\bar{w}$ and equate the accept/reject offers:

$$\begin{aligned}
\frac{\bar{w}}{1-\beta} =& c+\beta\int_0^{\bar{w}}\frac{\bar{w}}{1-\beta}dF(w')+\beta\int_{\bar{w}}^B\frac{w'}{1-\beta}dF(w') \qquad (8)\\
&\Rightarrow \frac{\bar{w}}{1-\beta}\int_0^{\bar{w}}dF(w')+\frac{\bar{w}}{1-\beta}\int_{\bar{w}}^B dF(w')=c+\beta\frac{\bar{w}}{1-\beta}\int_0^{\bar{w}}dF(w')+\frac{\beta}{1-\beta}\int_{\bar{w}}^B w'dF(w')\\
&\Rightarrow \bar{w}\int_0^{\bar{w}}dF(w')-c=\frac{1}{1-\beta}\int_{\bar{w}}^B(\beta w'-\bar{w})dF(w')\\
&\Rightarrow (\bar{w}-c)=\frac{\beta}{1-\beta}\int_{\bar{w}}^B(w'-\bar{w})dF(w') \qquad \text{(After adding } \bar{w}\textstyle\int_{\bar{w}}^B dF(w') \text{ to both sides)}
\end{aligned}$$

The lefthand side of this equation is the (opportunity) cost of searching one more time when the offer of $\bar{w}$ is in hand (i.e. you will earn c by rejecting when you could have earned $\bar{w}-c$ more if you had accepted). The righthand side of the equation is the expected benefit of searching one more time. $\frac{\beta}{1-\beta}$ represents a one-period discount + the infinite discounted sequence of accepting the offer next period. Lastly, the term $\int_{\bar{w}}^B(w'-\bar{w})dF(w')$ represents the expected surplus value of an accepted wage offer (i.e. the wage value in excess of the reservation wage).

If we examine the RHS of the equation, we can redefine it as some function of the current wage offer in hand:

$$h(w)=\frac{\beta}{1-\beta}\int_w^B(w'-w)dF(w'). \qquad (9)$$

What are its properties? We know $h(0)=\frac{\beta}{1-\beta}E[w]$ and that $h(B)=0$. $h(w)$ is differentiable with derivative[16]

$$h'(w)=-\frac{\beta}{1-\beta}[1-F(w)]<0.$$

It can also be verified that the second derivative is positive, suggesting a function with intercept $\frac{\beta E[w]}{1-\beta}$ and with a diminishing negative slope that tends to zero. Now, plotting the line $w-c$ and $h(w)$ one can obtain a unique solution at the intersection in the positive quadrant.

Another useful characterization comes from expanding the initial equation (8) to

$$\begin{aligned}
\bar{w}-c=&\frac{\beta}{1-\beta}\int_{\bar{w}}^B(w'-\bar{w})dF(w')+\frac{\beta}{1-\beta}\int_0^{\bar{w}}(w'-\bar{w})dF(w')-\frac{\beta}{1-\beta}\int_0^{\bar{w}}(w'-\bar{w})dF(w')\\
=&\frac{\beta}{1-\beta}E[w]-\frac{\beta}{1-\beta}\bar{w}-\frac{\beta}{1-\beta}\int_0^{\bar{w}}(w'-\bar{w})dF(w'),
\end{aligned}$$

re-written to

$$\bar{w}-(1-\beta)c=\beta E[w]-\beta\int_0^{\bar{w}}(w'-\bar{w})dF(w'),$$

and after applying integration by parts to the righthand side and rearranging some terms, we get

$$\bar{w}-c=\beta(E[w]-c)+\beta\int_0^{\bar{w}}F(w')dw'. \qquad (10)$$

Once again,we obtain a not so intuitive expression on the righthand side for the expected benefit of searching one more period. Let's define the function

$$g(s)=\int_0^s F(p)dp \qquad (11)$$

and examine its properties. In particular $g(0)=0, g(s)\geq 0, g'(s)=F(s)>0$ and $g''(s)=F'(s)=f(s)>0$ for all s. Once again, we can equate the two equations to find a solution for the reservation wage. This solution

[16]Use Leibniz's rule: Let $\phi(t)=\int_{\alpha(t)}^{\beta(t)}f(x,t)dx$. Then $\phi(t)'=f(\beta(t),t)\beta'(t)-f(\alpha(t),t)\alpha'(t)+\int_{\alpha(t)}^{\beta(t)}f_t(x,t)dx$.

$\bar{w}$ is the same solution obtained form the previous formulations, of course. What can be taken away from this setup? First, an increase in unemployment compensation will increase the reservation wage. The increase in c will shift both equations down but both are characterized by positive slopes where the RHS of (10) is increasing in w; thus, the new equilibrium reservation wage will be larger (i.e. $\bar{w}(c + \Delta c) \geq \bar{w}(c)$. Secondly, holding unemployment compensation constant, a mean-preserving increase in risk causes $\bar{w}$ to increase, as well. This quality can be proved by looking at (11) and recalling properties of a mean-preserving spread. In particular, (5) shows that if r_2 has been obtained by a mean-preserving spread over r_1 then taking the RHS of the cost/benefit equation (10) we observe

$$\beta(E[w] - c) + \beta \int_0^{\bar{w}} F(w', r_2)dw' - \left[\beta(E[w] - c) + \beta \int_0^{\bar{w}} F(w', r_1)dw'\right]$$
$$\Rightarrow \beta \int_0^{\bar{w}} [F(w', r_2) - F(w', r_1)]dw \geq 0$$

by definition for all possible values in the support. Thus, an increase in risk leads to a positive shift in the curve, representing expected benefit of search. Ultimately, this leads to an increase in the reservation wage. What insights could this possibly offer? Wages are bounded below by 0 and workers have unemployment compensation c to fall back on. Therefore, an increase in risk generally means more volatility in wage offers in the positive direction, given the bound. Knowing this, a worker is more inclined to wait another period and see if an exceptionally high job offer is made.[17]

Given this model setup, it is easy to calculate the probability of waiting some length of time until a job offer is accepted. In particular, define N as *length of time* until an offer is accepted. Given a reservation wage, a worker rejects an offer with probability $\lambda = \int_0^{\bar{w}} dF(w')$ in any given period, and accepts with probability $(1 - \lambda)$. Thus, we can predict waiting times according to a geometric probability distribution. The probability of waiting N periods until acceptance is simply $\lambda^{N-1}(1-\lambda)$, and the average wait time for a worker until accepting an offer is $(1-\lambda)^{-1}$.[18]

Lastly, it may be of benefit to allow for the possibility that workers can be fired; thus, accepting a wage offer does not ensure a lifetime income stream of w. Thus, after accepting a job in period t, the worker now faces a probability α of being fired in every subsequent period. If fired, the worker is not allowed to search in that period and instead receives unemployment compensation c. In the following periods, we are back to where we started: the worker fields job offers and decides whether to accept or continue searching. The Bellman equation would then become

$$\hat{v}(w) = \max_{\text{accept,reject}} \{w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha[c + \beta E[\hat{v}], c + \beta E[\hat{v}]\},$$

where $E[\hat{v}] = \int \hat{v}(w)dF(w)$. Under the reservation wage setup, we get

$$\hat{v}(w) = \begin{cases} \frac{w+\beta\alpha[c+\beta E[\hat{v}]]}{1-\beta(1-\alpha)}, & \text{if } w \geq \bar{w} \\ c + \beta E[\hat{v}], & \text{if } w \leq \bar{w}, \end{cases}$$

[17] What if the worker was allowed to quit his job? If this were allowed in the game, little would change in the way of rational decision-making. To see, consider the fact that there are now three scenarios: 1) accept the wage and keep job forever, 2) accept the wage but quit after t periods, or 3) reject the wage and continue the job search. By drawing out the respective Bellman equations, one can see that a worker would never prefer the second scenario. Thus, its inclusion in this model would only add unnecessary complexity. On the contrary though, people choose to quit jobs all of the time; thus, the model should find ways to incorporate such outcomes.

[18] Note that these qualities depend heavily upon the fact that the reservation wage is fixed throughout time. Would these results change if we diverged from an infinite horizon to a finite horizon environment? Also, what if randomness was incorporated into the worker's job preference and/or if the worker faced short-term shocks in demand for liquidity?

where $\bar{w}$ solves

$$\frac{\bar{w} + \beta\alpha[c + \beta E[\hat{v}]]}{1 - \beta(1-\alpha)} = c + \beta E[\hat{v}],$$

which can be rearranged as

$$\frac{\bar{w}}{1-\beta} = c + \beta \int \hat{v}(w')dF(w').$$

One of the main implications of this model is that the *firing* value function $\hat{v}(w)$ is strictly less than the $v(w)$ for all $w \in [0, B]$. Intuitively, no agent would rationally choose to quit a job in the standard model; thus, no agent would prefer to be fired from a job, once they have accepted a job offer. Given this relationship, the reservation wage $\hat{\bar{w}}$ is strictly lower than $\bar{w}$. Why? Workers are less likely to hold out for a higher/better job offer next period if they know it is more probable that they will not retain that job forever.

Consider an economy with a unit mass of agents, $i \in [0, 1]$, each of them that can be in any of the two states, employed or unemployed, $s \in \{E, U\}$. Then the Markov matrix Π is given by:

$$\Pi = \begin{pmatrix} 1-\alpha & \alpha \\ 1-\lambda & \lambda \end{pmatrix}$$

The invariant distribution, μ^*, which is the fraction of time an agent i spend in each state. If we let μ_t be the fraction of employed agents (and thus $1 - \mu_t$ the fraction of unemployed people), then the employment rate can be calculated with the following formula:

$$\mu_{t+1} = \mu_t(1-\alpha) + (1-\mu_t)(1-\lambda)$$

Then the invariant distribution, $\mu_t = \mu_{t+1} = \mu^*$ is given by $\mu^* = \dfrac{1-\lambda}{\alpha + 1 - \lambda}$.

5.3 A Prelim Example

Setup: If previously unemployed, a worker enters a period with a wage offer w in hand, where w is independently drawn from the distribution $F(w)$. If the worker refuses the offer, she receives compensation b for the period. If the offer is accepted, the worker receives w as an input to her strictly increasing utility function $u(w_t)$, discounted at a rate $\beta \in (0, 1)$. If working, there is a probability δ the worker will not be fired in the following period.

Assumption 5.1. *Worker utility function is $u(w) = w$.*

Assumption 5.2. *Wage offers are bound in the range $w \in [0, W]$.*

Assumption 5.3. *If fired in period t, the worker receives compensation b and can search for offers next period.*

Recursive Formulation: Unemployment U is characterized as

$$\begin{aligned} U =& b + \beta \int_0^W max\{V(w), U\} dF(w) \qquad (1) \\ =& b + \beta EJ \end{aligned}$$

where $J(w) = max\{V(w), U\}$ and $V(w)$ is the value of employment with offer w in hand, represented as

$$V(w) = w + \beta[\delta V(w) + (1-\delta)U]. \tag{2}$$

Together, equations (1) and (2) characterize the worker's problem as a dynamic program.

Existence and Uniqueness: We need to prove that there exists a unique reservation wage R, and that there is a unique solution to the dynamic program, described above. First, another assumption.

Assumption 5.4. *$V(0) < U < V(W)$.*

If this does not hold, then nothing interesting can really be said about the economic environment: the worker always accepts a job offer if $U \leq V(0)$, and the worker never accepts a job offer if $U \geq V(W)$.

Claim 5.1. *$V(W)$ is strictly increasing.*

Proof. Using equation (2),

$$V'(w) = 1 + \beta\delta V'(w) \Rightarrow V'(w) = \frac{1}{1-\beta\delta} > 0,$$

given that $\delta \in (0,1)$ and $\beta \in (0,1)$. □

Given that U is a constant value, V is strictly increasing and and the assumption that $U \in (V(0), V(W))$, there exists a unique R such that $V(R) = U$, where

$$\begin{aligned} V(R) = R + &\beta[\delta V(R) + (1-\delta)U] \\ &\Rightarrow V(R) = \frac{R + \beta(1-\delta)U}{1-\beta\delta}, \end{aligned}$$

such that we can isolate the value of U by expanding the identity $V(R) = U$:

$$\begin{aligned} &\frac{R + \beta(1-\delta)U}{1-\beta\delta} = U \\ &\Rightarrow R + \beta(1-\delta)U = (1-\beta\delta)U \\ &\Rightarrow U[1 - \beta\delta - \beta + \beta\delta] = R \\ &\Rightarrow U = \frac{R}{1-\beta}. \end{aligned} \tag{3}$$

Thus, there exists a reservation wage form on $J(w)$ such that

$$J(w) = \begin{cases} V(w), & \text{if } w \geq R. \\ \frac{R}{1-\beta}, & \text{if } w < R. \end{cases}$$

We'd like to prove the existence and uniqueness of the functional form of V. To do so, sub equation (1) into equation (2) to get

$$\begin{aligned} V(w) =& w + \beta\delta V(w) + \beta(1-\delta)(b + \beta \int_0^W max\{V(w), u\} dF(w)) \\ =& \Delta + \frac{\beta^2(1-\delta)}{1-\beta\delta} \int_0^W max\{V(w), U\} dF(w), \end{aligned}$$

with $\Delta = \frac{w+\beta(1-\delta)b}{1-\beta\delta}$. Now we can define the operator $T : B(w) \to B(w)$ on the space of bounded functions by

$$T(V(w)) = \Delta + \frac{\beta^2(1-\delta)}{1-\beta\delta} \int_0^W max\{V(w), U\} dF(w). \tag{4}$$

Claim 5.2. *The operator T as defined in equation* (4) *satisfies Blackwell's sufficient conditions, and is thus a contraction with modulus β.*

Proof. First, we check for *discounting*. Given the constant $a \in \mathbb{R}$,

$$\begin{aligned} T(V(w) + a) =& \Delta + \frac{\beta^2(1-\delta)}{1-\beta\delta} \int_0^W max\{V(w) + a, U\} dF(w) \\ \leq& \Delta + \frac{\beta^2(1-\delta)}{1-\beta\delta} \int_0^W max\{V(w) + a, U + a\} dF(w) \\ =& \Delta + \frac{\beta^2(1-\delta)}{1-\beta\delta} \int_0^W max\{V(w), U\} dF(w) + \frac{\beta^2(1-\delta)}{1-\beta\delta} a \\ \leq& \Delta + \frac{\beta^2(1-\delta)}{1-\beta\delta} \int_0^W max\{V(w), U\} dF(w) + \beta a \\ =& T(V(w)) + \beta a. \end{aligned}$$

Hence, the operator T satisfies the discounting property. Now, we check for *monotonicity*. Let $f(w), g(w) \in B(w)$ such that $f(w) \geq g(w) \quad \forall w \in [0, W]$. Then, we observe

$$\begin{aligned} T(f(w)) - T(g(w)) =& \frac{\beta}{1-\beta\delta} \left[\int_0^W max\{f(w), U\} dF(w) - \int_0^W max\{g(w), U\} dF(w) \right] \\ =& \frac{\beta}{1-\beta\delta} \left[\int_0^W max\{f(w) - g(w), 0\} dF(w) \right] \\ \geq& \ 0 \end{aligned}$$

and this holds because $f(w) \geq g(w)$ for all possible offers of w. Hence, the operator T satisfies the monotonicity property, implying that T is a contraction with modulus β. □

As a contraction, we know that there exists a unique solution V to the dynamic program with a corresponding unique reservation wage R.

<u>R **Increasing in** b**:**</u> Now, we are going to use results and properties of the environment to pin down the relationship between unemployment compensation b and the reservation wage R. Start with the initial formula for $U = b + \beta EJ$:

$$\begin{aligned}
(1-\beta)U =& b - \beta U + \beta \int_0^W max\{V(w), U\} dF(w) && \text{(substract by } \beta U) \\
=& b + \beta \int_R^W [V(w) - U] dF(w) \\
=& b + \beta \Big[(V(w) - U)F(w) \Big|_R^W - \int_R^W V'(w)F(w)dw \Big] && \text{(int. by parts)} \\
=& b + \beta \Big[V(W) - U - \int_W^R V'(w)F(w)dw \Big] && (\text{bc } V(R) = U) \\
=& b + \Big[\frac{W + (\beta - 1)U}{1 - \beta\delta} - \int_R^W V'(w)F(w)dw \Big] && (V(W) = \tfrac{W + \beta(1-\delta)U}{1-\beta\delta}) \\
=& b + \Big[V'(w)[W + (\beta - 1)U] - \int_R^W V'(w)F(w)dw \Big] && (V'(w) = \tfrac{1}{1-\beta\delta}) \\
=& b + \beta \Big[V'(w)(W - R) - \int_R^W V'(w)F(w)dw \Big] && (U = \tfrac{R}{1-\beta}) \\
=& b + \beta \int_R^W V'(w)[1 - F(w)]dw.
\end{aligned}$$

Thus, we know the following relationship must hold:

$$R = b + \phi(R) \tag{5}$$

where $\phi(R) = \beta \int_R^W V'(w)[1 - F(w)]dw$ and we once again make use of the fact that $U = \frac{R}{1-\beta}$. By Leibniz's rule, we have

$$\phi'(R) = -V'(R)[1 - F(R)] < 0.$$

Claim 5.3. *R is (weakly) increasing in b.*

Proof. Suppose not. Re-organize (5) as $b = R - \phi(R)$. If b increases, then it must be that $R - \phi(R)$ is strictly decreasing. This contradicts the relationship of equation (5); hence, it must be that R is weakly increasing in b. □

6 Recursive Competitive Equilibrium

When discussing equilibrium concepts for the neoclassical growth model in sections 2 and 4, we focused on two distinct methods that turned out to be equivalent under the right conditions. In particular, we looked at the sequence problem, whereby the time 0 agent chooses a sequence of allocations, given price information. We also looked at the functional equation, whereby the the infinite dimensionality of the sequence problem was reduced to a *two period* model of sorts, which was embodied in the Bellman equation representation. It is towards this latter solution concept that recursive competitive equilibrium (RCE) is related to. *Recursion* refers to the repeated application of a rule or procedure, and that is exactly what is accomplished here. In particular, for any given period, we define the *state* of the economy by a set of objects which completely characterize all relevant information for making decisions. Using the agent's *value* and *policy functions*, decisions can optimally be made with respect to current and future utility, one period at a time. In addition, laws of motion for relevant variables track and determine the changes in the aggregate state variables. This is recursive in nature because the value and policy functions are time invariant and simply take current states as their inputs. This discussion is adapted from Professor Chari's lectures. For additional reference—and a slightly different treatment—you can refer to Ljunqgvist and Sargent Chapters 7,8 and 12.

6.1 Complete Markets

The rest of this section has been replaced with your notes from Chari's class this year. I edited it for a few typos, comments, etc.

Consider a stochastic Arrow-Debreu problem for the household:

$$\max \quad \sum_{t=0}^{\infty}\sum_{s^t}\mu_t(s^t)u(c_t(s^t),l_t(s^t)) \tag{AD}$$

$$s.t. \sum_{t=0}^{\infty}\sum_{s^t}p_t(s^t)(c_t(s^t)+k_{t+1}(s^t)-(1-\delta)k_t(s^{t-1})) \leq \sum_{t=0}^{\infty}\sum_{s^t}p_t(s^t)(w_t(s^t)l_t(s^t)+r_t(s^t)k_t(s^{t-1}))$$

$$s.t. \qquad \text{non-negativity constraints}$$

From a mechanical standpoint, all the activity in this economy occurs at date 0: the consumer faces a maximization problem with a single lifetime budget constraint. Essentially, households take stock of future states of the world and make decisions about future consumption, labor and investment in the initial time period. This representation does not capture the normal way in which we imagine interaction in an economy; thus, we consider the sequential market problem and derive an equivalence between the two. The sequential markets problem is stated with a period-by-period budget constraint

$$\max \quad \sum_{t=0}^{\infty}\sum_{s^t}\mu_t(s^t)u(c_t(s^t),l_t(s^t)) \tag{SMP}$$

$$s.t. \quad c_t(s^t)+k_{t+1}(s^t)-(1-\delta)k_t(s^{t-1}) \leq w_t(s^t)l_t(s^t)+r_t(s^t)k_t(s^{t-1}) \quad \forall t, \forall s^t$$

$$s.t. \quad \bar{K} \geq k_{t+1}(s^t) \geq 0 \quad \forall t, \forall s^t$$

where we seek to find the conditions such that the two representations are the same.

Definition 6.1. *A Sequential Markets equilibrium is an allocation* $\{c_t(s^t), l_t(s^t), k_{t+1}(s^t)\}$ *and price system* $\{r_t(s^t), w_t(s^t)\}$ *such that*

i. *Households solve the SMP*

ii. *Firms maximize; that is,*

  a. $A(s^t)F_k(k_t(s^{t-1}), l_t(s^t)) = r_t(s^t)$

  b. $A(s^t)F_l(k_t(s^{t-1}), l_t(s^t)) = w_t(s^t)$

iii. *And markets clear.*

6.1.1 A Simplified Example

To help motivate and prove the equivalence between the Arrow-Debreu setup and the Sequential Markets representation, we will consider a simple example. Let the example economy be deterministic and have an infinite horizon with a finite number of types of agents. Let there be I types and assume that there is an equal mass of each type. Each agent is endowed with e_t^i for each period t which implies resource feasibility

$$\sum_i c_t^i = \sum_i e_t^i \tag{1}$$

Thus, an allocation consists of choosing a consumption sequence $\{\{c_t^i\}_t\}_i$ in solving the household problem

$$\max \sum_{t=0}^{\infty} u_i(c_t^i) \tag{2}$$

$$s.t. \quad \sum_t p_t c_t^i \leq \sum_t p_t e_t^i$$

Definition 6.2. *An Arrow-Debreu equilibrium is defined as a consumption allocation* $\{\{c_t^i\}_t\}_i$ *and a price system* $\{p_t\}$ *such that*

1. *Households solve (2), and*

2. *Resource feasibility (1) is satisfied.*

Now, consider the period-by-period counterpart to problem (2) along with a new price q_t:

$$\max \quad \sum_{t=0}^{\infty} u_i(c_t^i) \tag{3}$$

$$s.t. \quad c_t^i + q_t a_{t+1}^i \leq e_t^i + a_t^i \quad \forall t$$

where q_t is the period t price of a one-period, discount bond and a_{t+1}^i is a bond that pays 1 unit of consumption to agent i in period $t+1$, which was purchased in period t.[19] At this point, the problem is <u>not</u> well-defined. In particular, the agent has incentives to boost consumption by continually raising his debt and doing so to an arbitrarily large limit. Thus, to keep the problem bounded, we include a No-Ponzi condition

$$a_{t+1}^i \geq -\bar{A} \qquad \text{with } \bar{A} \in \mathbb{R}_+ \tag{4}$$

[19] If the return on such assets is 1 and the price paid is q_t, the gross interest rate R_t can be computed as $1/q_t$.

which places an *ad hoc* lower bound on the level of debt the household is allowed to accrue. In a more realistic setting, we could replace condition (4) with a *natural debt limit* which derives the bound $\bar{A}$ as the maximal level of debt such that the consumer still could feasibly repay it over time.

Theorem 6.1. ***(AD → SM)*** *In the Arrow-Debreu economy, assume prices satisfy the boundedness condition* $\sum_{t=0}^{\infty} p_t < \infty$. *Then, there exists a sequence of assets* $\{a_{t+1}^i\}$, *prices* $\{q_t\}$ *and some bound* $\bar{A}$ *such that solution to the Arrow-Debreu problem is also the solution to the Sequential Markets problem.*

This theorem states that the sequences of consumption in both problems will be identical, which further implies that the maximized utility value will be the same. Now, for a theorem stating the reverse direction.

Theorem 6.2. ***(SM → AD)*** *Let there exists a* T *and an* $\epsilon > 0$ *such that*

$$q_t \leq \frac{1}{1+\epsilon} \qquad \forall t \geq T$$

and the No-Ponzi conditions holds with strict inequality, then any solution to the Sequential Markets problem is a solution to the Arrow-Debreu problem.

To understand the equivalence, consider the first-order condition of the AD problem

$$[c_t^i]: \quad \beta^t u_i'(c_t^i) = \lambda p_t \tag{5}$$

where λ is the multiplier on the time-zero budget constraint. Alternatively, for the sequential markets, consider the first-order conditions

$$[c_t^i]: \quad \beta^t u_i'(c_t^i) = \mu_t \tag{6}$$

$$[a_{t+1}^i]: \quad q_t \mu_t = \mu_{t+1} \tag{7}$$

Combine equations (2) and (3) to get

$$q_t = \beta \frac{u_i'(t+1)}{u_i'(t)}$$

and combine this result with (1) to arrive at the expression

$$q_t = \frac{p_{t+1}}{p_t} \tag{8}$$

which is an intuitive result. Price p_t represents the numeraire price of a unit of consumption in period t. When in period t, what price are you willing to pay for a unit of consumption in period $t+1$? You will pay p_{t+1}/p_t which is also the period-to-period price q_t.[20]

[20] The Arrow-Debreu $\{p_t\}$ prices are relative prices; thus, we can set the price $p_0 = 0$ as a common normalization. In addition, we can represent the relationship (8) as

$$p_0 = 1$$
$$p_1 = q_0$$
$$p_2 = q_0 q_1$$
$$\vdots$$

which guarantees that $\sum_{t=0}^{\infty} p_t < \infty$ is finite if and only if $\prod_{t=0}^{\infty} q_t < \infty$.

Next, how do we recover the assets $\{a^i_{t+1}\}$ for the model, given that we have pinned down the prices $\{q_t\}$? This is accomplished by summing forward the period budget constraint:

$$\begin{aligned} \lim_{T\to\infty} p_{T+1}a_{T+1} + \lim_{T\to\infty}\sum_{s=t}^{T} p_s(c^i_s - e^i_s) &= p_t a^i_t \\ \Rightarrow a^i_t &= \sum_{s=t}^{\infty}\frac{p_s}{p_t}(c^i_s - e^i_s) \end{aligned} \tag{9}$$

where the first term of the first line is equal to zero as long as assets are bounded and the p_T goes to zero in the limit. Consider equation (5). Divide by the time 0 condition with $p_0 = 0$ to get

$$p_t = \beta^t \frac{u'_i(c^i_t)}{u'_i(c^i_0)}$$

Given a discount factor $\beta < 1$ and a strictly concave utility function, interior solutions are guaranteed and the RHS should converge to zero in the limit.

6.2 Recursive Equilibria in the Neoclassical Growth Model

Now, let's examine some of the implications for the growth model

$$\max \quad \sum_{t=0}^{\infty}\sum_{s^t}\mu_t(s^t)u(c_t(s^t), l_t(s^t)) \tag{10}$$

$$s.t. \quad \sum_{t=0}^{\infty}\sum_{s^t} p_t(s^t)(c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1})) \le \sum_{t=0}^{\infty}\sum_{s^t} p_t(s^t)(w_t(s^t)l_t(s^t) + r_t(s^t)k_t(s^{t-1}))$$

$$s.t. \quad \bar{K} \ge k_{t+1}(s^t) \ge 0$$

where we assume that the stochastic innovations $\{s_t\}$ come from a finite set and follow a Markov process. We can now recast the problem in (6) as the functional equation

$$\begin{aligned} V(k,K,s) = \quad & \max \quad \{u(c,l) + \beta\sum_{s'}\pi(s'|s)v(k',K',s')\} \\ & s.t. \quad c + k' - (1-\delta)k \le w(K,s)l + r(K,s)k \\ & s.t. \quad K' = G(K,s) \end{aligned} \tag{FE}$$

In this framework, K denotes the aggregate stock of capital in the economy whereas k represents the households individual holding of the capital stock. Why is it necessary to keep track of the aggregate capital stock? The answer is two-fold:

1. We need prices to be determined competitively in the economy. This implies that no particular household's activity can influence the price level. You can think of the FE as a household's problem in which that household is just one of infinitely many on the unit interval. As such, the household chooses quantities taking prices and aggregates as given.

2. In order to solve this problem, the household needs to be able to forecast future prices, which depend upon

the decisions of everyone in the economy. This leads to an equilibrium price determination

$$w(K,s) = A(s)F_L(K,L)$$
$$r(K,s) = A(s)F_K(K,L)$$

given the production function $Y = AF(K,L)$.

In addition, notice that the FE problem is characterized by a law of motion G for the aggregate capital stock which households know and take as given. In solving the FE, we define the solutions as policy functions $c(k,K,s), l(k,K,s)$ and $g(k,K,s)$ for consumption, labor and next-period capital, respectively. Let the space of feasible capital be a constant set X and the random shocks come from the sigma-algebra Ω with a corresponding probability space.

Definition 6.3. *A Recursive competitive equilibrium is defined by a value function $v : X \times X \times \Omega \to \mathbb{R}$, policy functions $\{c, l, g\}$ of the same domain as v and prices $\{w(K,s), r(K,s)\}$ such that*

1. *v attains the maximum of the FE problem, given $G(\cdot)$,*
2. *the policy functions solve the FE problem,*
3. *Firms maximize profits:*

$$w(K,s) = A(s)F_L(K,l(K,K,s))$$
$$r(K,s) = A(s)F_K(K,l(K,K,s))$$

4. *g(K,K,s) = G(K,s)*

In processing this definition, take special notice of the consistency restrictions imposed on the policy functions, where they take aggregate capital as an input for their household level k in parts (3) and (4). Thus, while the household takes the aggregate as given, when in equilibrium and if endowed with the total quantity of capital, the household will choose the same levels of labor and capital that coincide with equilibrium prices and the aggregate law of motion.

7 Incomplete Markets and Heterogeneous Agent Models

7.1 The General Model

In this section, we will consider economies in which there exists a continuum of agents which are *ex ante* identical, but due to stochastic shocks and limitations in the asset/insurance markets, have *ex post* heterogenuous asset holdings. In a complete markets setting, whether in an Arrow-Debreu or Sequential Markets settings, agents are able to trade and hedge against various idiosyncratic shocks. For example, an agent can buy contingent claims on consumption next period, which depend upon whether or not a high or low endowment shock occurs. To do so, the agent must pay some price today (i.e. must pay some kind of premium). When borrowing is limited, as is the case with incomplete markets, agents must *self-insure*: they are left with stock-piling quantities of some asset. As one risk-averse person may expect, when times are good (shocks are positive), agents choose to accumulate a larger *rainy day fund* of assets, and when times are bad (shocks are negative), they are forced to consume their stockpile of assets in an attempt to smooth consumption over time.

In this model, the environment is characterized by an infinitely-lived unit measure of households with preference[21]

$$\mathrm{E}\sum_{t=0}^{\infty}\beta^t u(c_t). \tag{1}$$

We further assume that this is a simple endowment economy, where the agent receives a stochastic endowment $y_t \in Y$ every period. We will consider this distribution of assets to be finite; thus, we further assume that the probability is determined by the transition matrix $\pi(y'|y)$. In addition, a key and essential assumption that we make is no aggregate uncertainty. Thus, while individuals face idiosyncratic shocks that are not completely insurable, the economy has the same level of endowment in each period. For motivation and justification of this assumption, we can appeal to a law of large numbers result here, since there exists an infinity of agents on the unit interval.

Let $\Pi(y)$ be the unconditional stationary distribution of y and assue that $\Pi(y)$ also represents the fraction of households with endowment y. Therefore, the interpretation is twofold: i) $\pi(y)$ provides the unconditional probability of receiving endowment y and ii) the fraction of the households on the unit interval that receive y is $\Pi(y)$. As mentioned before, the aggregate endowment is constant at some value $\overline{y}$. Households are faced with the constraints

$$a' + c = y + (1+r)a, and \tag{2}$$

$$a' \geq -b, \tag{3}$$

where (2) is a standard budget constraint and (3) is a borrowing constraint. Equation (2) states that today's income (from both the stochastic endowment and the return on the agent's asset stock) must be divided between current consumption and next-period assets. Equation (3) states that the consumer may not choose assets below some debt limit $-b$. Thus, if an agent experiences a particular bad streak of endowment shocks and has a low current-period level of assets, it may be preferable to choose a large negative level of assets to help smooth consumption. The debt limit $-b$ may get in the way of allowing the agent to do so.[22]

[21] This section is in large part taken from the notes by Lutz Hendricks.

[22] More on natural debt limit and specifying this particular value.

Thus, choices depend upon the state vector (a, y), the rate of return on assets r and this decision problem may be formulated with the Bellman equation as

$$v(a, y) = \max_{c,a'} u(c) + \beta \sum_{y'} \pi(y'|y) v(a', y') \tag{4}$$

subject to the borrowing constraint and the debt limit constraint. Subbing the budget constraint into the utility function and choosing a lagrange multiplier μ for the debt limit we can form the lagrangian

$$\mathcal{L} = u(y + (1 + r)a - a') + \beta \sum_{y'} \pi(y'|y) v(a', y') + \mu(a' + b),$$

which yields the following first-order conditions

$$u'(c) = \beta \sum \pi(y'|y) \frac{\partial v(a', y')}{\partial a'} + \mu \tag{5}$$

$$\frac{\partial v}{\partial a} = u'(c)(1 + r) \tag{6}$$

$$\mu(a' + b) = 0, \mu \geq 0, a' + b \geq 0. \tag{7}$$

Equation (5) comes from differentiating with respect to the choice of next-period assets, while equation (6) is the envelope condition and equation (7) is a complementary slackness condition on choosing assets above the debt limit. If the debt limit is binding for consumers, then $\mu > 0$ and $u'(c) > E[\beta(1 + r)u'(c')]$. What does this mean? This shows that if the debt limit is a barrier to agents properly smoothing their consumption over time, then it reduces welfare to a level underneath that which would be attainable with complete markets. The solution concept for the household problem is the value function and policy functions for consumption and next-period asset holdings. Further, to characterize equilibrium, we must define a joint distribution of assets and endowments in the economy. We will use $\Phi(a_0, y_0) = Prob\{a \leq a_0, y \leq y_0\}$.

Definition 7.1. *A stationary equilibrium is a value function $v(a, y)$ and the policy functions $c(a, y)$ and $a'(a, y)$ for households, a price function $r(\Phi)$ and a joint distribution of assets/endowments $\Phi(a, y)$ such that*

1. *households optimize with respect to their constraints,*

2. *markets clear, and*

3. *Φ is a time-invariant distribution.*

We've already defined the households problem. For market clearing, we must observe

$$\int\int c(a, y)\Phi(da, dy) = \int y\Pi(dy) = \overline{y} \tag{8}$$

$$\int\int a'(a, y)\Phi(da, dy) = 0 \tag{9}$$

for each period. Let's look at equation (8). The RHS shows that expected value of aggregate endowments for any period. Given our law of large numbers result, we have assumed that his quantity is fixed at $\overline{y}$. The LHS shows aggregate consumption in the economy, summed over all possible levels of this-period asset holdings and all possible shocks. $c(a, y)$ is a deterministic function and the integration just sums up these policy functions for every possible combination of asset holdings and shock in the economy. Lastly, equation (9) assumes that the net aggregate level of assets is zero.[23]

Lastly, we turn to the time invariance of the joint distribution Φ. Technically, this is allowed to vary from day to day (i.e. from $\Phi(a, y)$ to $\Phi'(a', y')$). How? It is determined by the household policy functions and the past random shocks. In particular, since a' is determined in the period before, the policy functions $a'(a, y)$ will play a role in determining a law of motion for Φ. The equilibrium is stationary when $\Phi' = \Phi$. Let's define a transition function $Q((a, y), (A, Y))$. It's interpretation is the probability of ending up in state $(a', y') \in (A, Y)$ tomorrow, given that you find yourself in state (a, y), today. By assumption, this probability is also equal to the fraction of the population that will end up in (A, Y) tomorrow. Specifically, we write

$$Q((a,y),(A,Y)) = \begin{cases} \sum_{y' \in Y} \pi(y'|y), & \text{if } a'(a,y) \in A \\ 0, & \text{Otherwise ,} \end{cases} .$$

First, what is the probability of ending up in the set of shocks Y? It's the sum of the probabilities for ending up with $y' \in Y$, given the state of y today. In addition, what's the probability of ending up in the future shock set Y and with asset levels $a' \in A$? It's the same sum of probabilities, excluding the ones where the policy function $a'(a, y) \notin A$. $a'(a, y)$ is deterministic and chosen by the agent in the previous period, so obviously we do not want to count the states that don't have the policy functions in the set A. Now, we may formalize the law of motion for the joint distribution function as

$$\Phi'(A,Y) = \int \int Q((a,y),(A,Y))\Phi(da,dy). \tag{10}$$

In (10), we define the joint distribution of assets and shocks tomorrow (RHS) as simply the probability of ending up in any set (A, Y) tomorrow, summed over all levels of a and y today. Stationarity is simply the case when $\Phi'(A, Y) = \Phi(A, Y) \quad \forall (A, Y)$.[24]

[23] For every liability, there is a matching asset of the same value. Thus, in equilibrium the sum of these values must be zero.

[24] IF there does not exist an invariant joint distribution, then we may also define a non-stationary equilibrium. We must redefine the Bellman equation and asset policy function as $v(a, y, \Phi)$ and $a'(a, y)$. Further, the returns on assets are now a function of the aggregates in the economy, which fluctuate; so, $r(\Phi)$. Lastly, the law of motion for the joint distribution is now some potentially non-linear function of the previous state: $\Phi' = H(\Phi)$. Our equilibrium definition is the same, but now with the stationarity/time-invarince condition dropped.

7.2 An Application: The Huggett Model

This section presents the paper by Mark Hugget (1993) from the Journal of Economic Dynamics and Control, titled *The risk-free rate in heterogeneous-agent Incomplete-insurance Economies.* The prime motivation for the paper was the existence of risk-free rates and equity premiums that were too low and high, respectively, given the calibration of representative agent model economies. In this model, agents face uninsurable idiosyncratic endowment shocks and they smooth their consumption through holding a risk-free asset.

Consider an exchange economy with a continuum of agents with total mass equal to one. Each period, agents will receive an endowment of a perishable consumption good. For simplicity, the endowment is either high (e_h) or low (e_l). The set of endowment is denoted $E = \{e_l, e_h\}$. We will assume that agent's endowment follow a Markov process with a stationary transition probability $\pi(e'|e) > 0$, which is independent of all other agents' current and past endowments. Agent's have identical preferences, given by

$$E[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}], \tag{1}$$

where we assume $\sigma > 1$. Agents are allowed to hold some credit balance of a units, entitling an agent to a units of consumption in the respective period. If the agent wishes to receive a' units of the consumption good next period, he must pay qa' good this period where q denotes the price of next-period balances. We also assume that there exists a credit limit $\underline{a} < 0$ that agents can never go below. The agents budget constraint is

$$c + qa' \leq a + e \quad \text{with } c \geq 0 \text{ and } a' \geq \underline{a} \tag{2}$$

So, every period, the agent observes the sum of his endowment + the assets he received that period. From there, he chooses how to divide this revenue between current consumption and buying next-period assets. Assets can also be negative: thus, the consumer can consume in excess of his income $a + e$ by choosing negative next-period assets, meaning he will have to pay back those liabilities in the following period.

Now, we'd like to formalize the agent's problem in terms of a Bellman equation. An agent's position in time is denoted by a state vector $x \in X$ where $x = (a, e)$. Thus, all that an agent is required to know to make optimal decisions is his current level of assets and the current-period shock to his endowment stream. Thus, the individual state space is $X = A \times E$ where $A = [\underline{a}, \infty]$ and E is denoted as before, with $e_h > e_l$. Thus, a functional equation is defined by

$$v(x; q) = \max_{(c,a') \in \Gamma(x,q)} u(c) + \beta \sum_{e'} \pi(e'|e) v(a', e'; q) \tag{3}$$

where the correspondence is

$$\Gamma(x; q) = \{(c, a') : c + qa' \leq a + e; c \geq 0; a' \geq \underline{a}\}. \tag{4}$$

Thus, given the state of the economy and the consumers endowment and asset holdings, he must choose consumption and next-period assets in a way that maximizes expected lifetime utility. If the function v is the optimal value function, then we can define policy functions $c : X \times \mathbb{R}_{++} \to \mathbb{R}_+$ and $a : X \times \mathbb{R}_{++} \to A$ given that they are measurable, feasible and satisfy

$$v(x;q) = u(c(x;q)) + \beta \sum_{e'} \pi(e'|e)v(a(x;q), e'; q).$$

Warning: some abuse of notation is upcoming.[25] What do we know about all of the agents? We know that they are not the same. Thus, we must develop a probability measure, defined on the subsets of the individual state space. Let Ψ be such a measure on (X, β_X) where $X = [\underline{a}, \overline{a}] \times E$ and β_X the Borel $\sigma-$algebra. Thus, $\Psi(B)$ indicates the mass of agents whose individual state vectors lie in B. Intuitively, pick an interval of assets holdings and the high or low endowment level (or both), and Ψ will tell you what fraction of the population (of measure one) will be with those traits.[26] For this model, we will assume that the probability measure Ψ and asset prices q are unchanging in time. Thus, we utilize a stationary probability measure and are interested in stationary equilibria.[27]

Definition 7.2. *A stationary equilibrium for this economy is* $\{v, c(x), a(x), q, \Psi\}$ *where* v *is a solution to the functional equation and*

i. $c(x)$ *and* $a(x)$ *are optimal policy functions solving the value function, given* q*,*

ii. Markets clear:

1. $\int c(a,e)\Psi(dx) = \int e\Psi(dx)$
2. $\int a(x)\Psi(dx) = 0$

iii. Ψ *is a stationary probability measure for all* $B \in \beta_X$*.*

So, equilibrium requires that the agents optimize utility with respect to their choices over consumption and next-period assets. Further, aggregate consumption must equal the aggregate endowment in any given period and any given state, and the net sum of all assets must be equal to zero, as well.

Before, we compute the equilibrium, let us define a mapping T on the space $C(X)$ of bounded, continuous real-valued functions on the individual state space X. We write

[25] We have described an individuals state space as $X = A \times E = [\underline{a}, \infty)] \times \{e_l, e_h\}$. These are the possible values that may occur for an individual. In what follows, we will need to construct a *similar* type of measure for the economy, denoted $S = [\underline{a}, \overline{a}] \times \{e_l, e_h\}$ which is the interval of all possible asset values along with all possible levels of the endowment shock. From this set, we can generate its Borel $\sigma-$algebra. This will be used in calculating aggregate probability measures for the economy. While this is done in the paper and is technically correct, I am excluding the new notation and simply using the individual state space X. Thus, when x appears as an element in X, you can interpret it with the same meaning of the S state space notation.

[26] Remember: by assumption, this is the unconditional probability of having assets and an endowment shock in the state B and it is the fraction of the population that is also in that state!

[27] Keeping with the notation from the actual paper, we define a transition function $Q : X \times \beta_X \to [0,1]$. $Q(x, B) = Q((a,e), B = (a', e'))$ is the probability that an individual with state x will have an individual state vector lying B next period, just as before but with slightly different notation this time. Then Ψ is said to be stationary given that

$$\Psi(B) = \int P(x,B)\Psi(dx) \quad \forall B \in \beta_S.$$

The stationarity comes from the fact that the LHS is not $\Psi'(B)$ but instead the same joint distribution as today's $\Psi(B)$. Further, this holds for all possible joint occurrences of assets a' and endowments e' (i.e. for all possile B).

$$(Tv)(x;q) = \max_{(c,a')\in\Gamma} u(c) + \beta \sum_{e'} \pi(e'|e)v(a',e';q), \tag{5}$$

where we define iterations of this operator by T^n suh that $T^1v = Tv$ and $T^2v = T(T(v))$ and so on.

Theorem 7.1. *For $q > 0$ and $\underline{a} + e_l > q\underline{a}$, there exists a unique solution v(x;q) $\in$ C(X) to (3) and the mappings $T^n v_0$ converge uniformly to v as $n \to \infty$ for any initical guess $v_0 \in C(X)$. Further, v(x) is strictly increasing, strictly concave and continuously differentiable in a. Furthermore, there exist continuous optimal decision rules $c(x;q)$ and $a(x;q)$ such that $a(x,;q)$ is strictly increasing in a for $(x;q)$ such that $a(x;q) > \underline{a}$ (and nondecreasing for $a(x;q) \geq \underline{a}$).*

Proof. First, we wish to show that the operator T is a mapping from the space $C(X) \to C(X)$. Due to the Inada conditions on consumption, c^* will always occur in the interior of the budget constraint (i.e. $c^* > 0$ for all states x). Further, the objective $\frac{c^{(1-\sigma)}}{1-\sigma}$ is bounded above for all paramter values $\sigma > 1$ and is bounded below for non-negative inputs c. Thus, there exists some M such that $|v(x;q)| < M$ for all $x \in X$. Further, the objective function $u(c)$ is continuous for $c > 0$; therefore, $(Tv)(c) : C(X) \to C(X)$.

Further, the operator T is a contraction on the space of bounded, continuous function. Specify the sup norm metric. Under Blackwell's sufficient conditions, we can prove monotonicity and discounting. That is.

i. For any $\hat{v}, \tilde{v} \in C(X)$ where $\hat{v}(x) \geq \tilde{v}(x) \forall x \in X, (T\hat{v})(x) = max u(x) + \beta\hat{v}(x) \geq max u(x) + \beta\tilde{v}(x) = (T\tilde{v})(x)$.

ii. $[T(v + a)](x) = max u(x) + \beta v(x) + \beta a = (Tv)(x) + \beta a$.

Thus, T is a contraction with a fixed point $v^* \in C(X)$ such that $(Tv^*)(x) = v^*$. Further, for any initial guess $v_0 \in C(X)$, the sequence $T^n v_0$ will uniformly converge to the fixed point.

Next, note that the objective function $u(c)$ is continuous in its argument and that the correspondence Γ is continuous, nonempty and compact. Thus, $(Tv)(x)$ is a continuous function. Through strictly increasing and strictly concave properties of the objective function, the value function is strictly increasing and strictly concave. Further, for any a, holding a' constant, consumption is nondecreasing in a. Beneviste and Schenkman for continuously and differentiable value function. □

The next theorem states conditions under which, for given q, there exists a unique stationary probability measure Ψ and it gives a method for computing excess demand in the credit market. Let us now define a mapping W : $M(X) \to M(X)$, induced by a transition function Q where $M(X)$ is the space of probability measures on (X, β_X). This is given by

$$(W\Psi)(B) = \int Q(x, B)\Psi(dx) \quad \forall B \in \beta_X.$$

We have defined the operator W but this is just another way of stating the general law of motion for the joint distribution as $\Psi'(B) = \int Q(x, B)\Psi(dx)$ like we did in the introduction to this section.

Theorem 7.2. *If the conditions of the first theorem hold, $\beta < q$ and $\pi(e_h|e_h) \geq \pi(e_h|e_l)$, then there exists a unique stationary probability measure Ψ on (X, β_X) and, for any $\Psi_0 \in M(X), W^n\Psi_0$ weakly converges to Ψ.*

This theorem states conditions under which the probability distribution is stationary (end hence the equilibrium is stationary) and that the probability distribution is unique.

Proof. Here, we utilize Theorem 12.12 from SLP, which states that, given the set X, if transition function Q is monotone, satisfies the Feller property and $\exists c \in X, \epsilon > 0$ and $N \geq$ such that $P^N(\underline{a}, [c, \overline{a}]) \geq \epsilon$ and $P^N(\overline{a}, [\underline{a}, c]) \geq \epsilon$, then P has a unique invariant probability measure Ψ such that $W^n\Psi_0 \to \Psi$.

First, we may assume that X is an ergodic set, which implies that $\forall n$ and $\forall a, b, P^N(a, b) > 0$. Thus, this easily satisfies the last condition for non-negative probabilities.

To prove P is monotone, we must show that for any increasing function $(Tf)(x) = \int f(x')P(x, dx')$ is an increasing function. Thus, we must show that TF is increasing in initial state assets a and initial state assets e. Note that we have assumed $\pi(e_h|e_h) \geq \pi(e_h|e_l)$. This implies

$$\int f(x')P((a, e_h), (da', de'_h) \geq \int f(x')P((a, e_l), (da', de'_h),$$

holding a constant. Thus, the operator is increasing in positive endowment shocks. Next, note for $\tilde{a} > a$,

$$\int f(x')P((\tilde{a}, e), (da', de') \geq \int f(x')P((a, e), (da', de'),$$

holding endowments constant, because the policy functions $a(a, e)$ are strictly increasing in initial assets a. Thus, we have shown that the operator (Tf) is increasing in its arguments, which implies that P is in fact monotone.

Lastly, we must prove the Feller property which requires that $(Tf)(x) = \int f(x')P(x, dx')$ is a continuous function for any bounded and continuous function f. Let us take any sequence of initial states $\{x_n\}$ in the state space X where $x_n \to x \in X$. Thus, we have $(Tf)(x_n) = \int f(x')P(x_n, dx')$. Apply limits

$$\begin{aligned} lim(Tf)(x_n) =& lim \int f(x')P(x_n, dx') \\ =& \int f(x') lim p(x'|x_n) d\lambda \\ =& \int f(x')P(x, dx') && \text{(Lebesgue Dominated Convergence Theorem)} \\ =& (Tf)(x) \end{aligned}$$

where the second line re-expresses the transition function as a conditional distribution. Thus, the operator $(Tf)(x)$ is a continuous function, satisfying the Feller property. Given that these conditions are satisfied, Ψ^n uniquely converges to an invariant joint distribution Ψ. □

If this distribution weakly converges then we can expect the sequence of asset holdings to converge; that is, we expect to observe

$$\int a(x; q)\Psi_n d(x) \to \int a(x; q)\Psi(dx),$$

which is expected to equal zero in equilibrium.

To calibrate this model to the real economy, we need to find parameter estimates for the set $(e_h, e_l, \sigma, \beta, \underline{a}, \pi(e_h|e_h), \pi(e_h|e_l))$ within the context of the proper period length. We will assume the low and high endowments as income when unemployed and employed, respectively. Further, we will consider a model in which six periods is interpreted as one calendar year. Reference the paper for more details as to the data collection and calibration, but we will use $e_h = 1, e_l = 0.1, \pi(e_h|e_h) = 0.925, \pi(e_h|e_l) = 0.5, \beta = 0.99322, \sigma = 1.5$ and range of possible credit limit levels $\underline{a} \in \{-2, -4, -6, -8\}$. Computation for equilibria proceeds as follows

1. Given price q, compute $a(x;q)$, given Theorem 1.
2. Given $a(x;q)$, iterate on $\Psi_{n+1}(B) = \int_s P(x,B)d\Psi_n$ from some arbitrary $\Psi_0 \in M(S)$. When the sequence has approximately converged, use probability measure to compute $\int_S a(x;q)d\Psi$.
3. Update q and repeat steps 1 and 2 until market clearing for the asset market is approximately observed.

What are the results from this paper after computing the equilibrium, given the parameter estimates? 1) The risk-free rate decreases as the credit limit increases. That is, as the credit limit is increased, the interest rate must be lowered to persuade agents to not overaccumulate credit balances so that the credit market can clear. 2) Higher risk aversion reduces the risk-free rate for all credit limits examined. We have found that with no aggregate uncertainty but with idiosyncratic shocks that are not perfectly insurable, the estimated risk-free rate is lower than a standard representative agent model with the same aggregate fluctuations. These results are closer to that observed in reality.

7.3 Arrow's Theorem

Consider an economy with $i = 1, 2, \ldots, I$ agents that have preferences over a single good given by a utility function, $u_i(.)$ and discount utility utility with β.

In each period t there is a realisation of a stochastic event $s_t \in S$. Let the history of events until time t be denoted $s^t = [s_0, s_1, \ldots, s_t]$. The unconditional probability of observing a particular sequence of events, s^t is given by a probability measure $\pi_t(s^t)$. Assume that $\pi_0(s_0) = 1$.

Agent i owns a stochastic endowment of the good $y_t(s^t)$ that depend on the history s^t.

Denote by $q_t^0(s^t)$ as the price of one unit of consumption where, the superscript 0 refers to the date when trades occur, while the subscript t refers to the date when deliveries are made.

Definition 7.3. *An Arrow-Debreu competitive equilibrium is a sequence of prices* $\{q_t^0(s^t)\}_{t=0}^{\infty}$ *and allocations* $\{(c_{it}(s^t))_{i=1}^{I}\}_{t=0}^{\infty}$ *for every history* s^t *such that:*

1. *Given prices* $\{q_t^0(s^t)\}_{t=0}^{\infty}$, *consumer* $i = 1, 2, \ldots, I$ *chooses* $\{(c_{it}(s^t))\}_{t=0}^{\infty}$ *to solve:*

$$\max_{\{c_{it}(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u_i(c_{it}(s^t))\pi_t(s^t)$$

subject to

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_{it}(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_{it}(s^t)$$

$$c_{it}(s^t) \geq 0 \qquad \forall t, s^t$$

2. *Markets clear:*

$$\sum_{i=1}^{I} c_{it}(s^t) = \sum_{i=1}^{I} y_{it}(s^t) \qquad \forall t, s^t$$

Denote by $\tilde{Q}_t(s_{t+1}|s^t)$ (the pricing kernel) as the price of one unit of time $t+1$ consumption, contingent on the realization s_{t+1} at $t+1$, when the history at t is s^t.

Definition 7.4. *A sequential markets equilibrium is a sequence of pricing kernel* $\tilde{Q}_t(s_{t+1}|s^t)$, *and allocations* $\{(\tilde{c_{it}}(s^t), \tilde{a_{it+1}}(s^{t+1}))_{i=1}^{I}\}_{t=0}^{\infty}$ *such that:*

1. *Given the pricing kernel,* $\{\tilde{Q}_t(s_{t+1}|s^t)\}_{t=0}^{\infty}$, *initial distribution of wealth* $\{\tilde{a}_{i0}(s_0)\}_{i=1}^{I}$ *and, the natural debt limit,* $\{A_{it+1}(s^{t+1})\}_{t=0}^{\infty}$, *consumers* $i = 1, 2, \ldots, I$ *choose* $\{\tilde{c}_{it}(s^t), \tilde{a}_{it+1}(s^{t+1})\}_{t=0}^{\infty}$ *to solve:*

$$\max_{\{\tilde{c_{it}}(s^t), \tilde{a}_{it+1}(s^{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u_i(c_{it}(s^t))\pi_t(s^t)$$

subject to

$$c_{it}(s^t) + \sum_{s_{t+1}} \tilde{a}_{it+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) \leq y_{it}(s^t) + \tilde{a}_{it}(s^t) \qquad \forall t, s^t$$

$$c_{it}(s^t) \geq 0 \qquad \forall t, s^t$$

$$\tilde{a}_{it+1}(s^{t+1}) \geq -A_{it+1}(s^{t+1}) \qquad \forall t, s^t$$

2. *Markets clear:*

$$\sum_{i=1}^{I} c_{it}(s^t) = \sum_{i=1}^{I} y_{it}(s^t) \qquad \forall t \geq 0, s^t$$

$$\sum_{i=1}^{I} \tilde{a}_{it+1}(s_{t+1}) = 0 \qquad \forall t \geq 0, s_{t+1}$$

Theorem 7.3. *Let* $\{\{c_{it}(s^t)\}_{i=1}^{I}\}_{t=0}^{\infty}, \{q_t^0(s^t)\}_{t=0}^{\infty}$ *constitute an Arrow-Debreu Equilibrium as defined in part a), then there exists an initial distribution of wealth* $\{\tilde{a}_{i0}(s_0)\}_{i=1}^{I}$, *sequence of natural debt limits and a corresponding sequential markets equilibrium as defined in part b), with allocations* $\{\{\tilde{c}_{it}(s^t), \tilde{a}_{it+1}(s^{t+1})\}_{i=1}^{I}\}_{t=0}^{\infty}$ *and pricing kernel* $\{\tilde{Q}_t(s_{t+1}|s^t)\}_{t=0}^{\infty}$ *such that:*

$$c_{it}(s^t) = \tilde{c}_{it}(s^t) \qquad \forall i, t, s^t$$

Proof.

By making an appropriate guess of the form of pricing kernel, it is easy to show that a competitive equilibrium allocation of the Arrow-Debreu model is also an allocation for a competitive equilibrium under sequential markets with a particular distribution of initial wealth. Take an initial guess of $\tilde{Q}_t(s_{t+1}|s^t)$ as follows:

$$q^0_{t+1}(s^{t+1}) = \tilde{Q}_t(s_{t+1}|s^t)q^0_t(s^t)$$

This implies that $\tilde{Q}_t(s_{t+1}|s^t) = q^t_{t+1}(s^{t+1})$ since, $q^t_{t+1}(s^{t+1}) = \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)}$.
The one-period Euler equation for the Arrow-Debreu economy is given by:

$$\frac{\beta u_i'(c_{it+1}(s^{t+1}))\pi_{t+1}(s^{t+1}|s^t)}{u_i'(c_{it}(s^t))} = \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)}$$

The one-period Euler equation for the sequential markets economy is given by:

$$\frac{\beta u_i'(\tilde{c}_{it+1}(s^{t+1}))\pi_{t+1}(s^{t+1}|s^t)}{u_i'(\tilde{c}_{it}(s^t))} = \tilde{Q}_t(s_{t+1}|s^t)$$

If our guess of the pricing kernel is correct, then the CE allocations under the Arrow-Debreu equilibrium are supported by the sequential market structure as shown above. The only thing remaining for us to choose is the initial distribution of wealth $\{\tilde{a}_{i0}(s_0)\}$ so that sequential market equilibrium duplicates the Arrow-Debreu competitive equilibrium allocation.
Claim: The initial distribution of wealth $\{\tilde{a}_{i0}(s_0)\}_{i=1}^{I}$ has to be zero for $i = 1, 2, \ldots, I$
Proof: For convenience, we first define the household's wealth, in terms of the date t, history s^t of the consumption good as:

$$\Upsilon_{it}(s^t) = \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t} q^t_\tau(s^t)[c_{it}(s^t) - y_{it}(s^t)]$$

The proof is once again by guessing that the household i chooses a portfolio given by $\tilde{a}_{it+1}(s_{t+1}, s^t) = \Upsilon_{it+1}(s^{t+1})$. Thus, the value of the portfolio expressed in terms of the date t, history s^t of the consumption good is given by:

$$\sum_{s_{t+1}} \tilde{a}_{it+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}, s^t) = \sum_{s^{t+1}|s^t} \Upsilon_{it+1}(s^{t+1})q^t_{t+1}(s^{t+1})$$

Substituting in the expression for $\Upsilon_{it+1}(s^{t+1})$, we get:

$$\sum_{s_{t+1}} \tilde{a}_{it+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}, s^t) = \sum_{\tau=t+1}^{\infty}\sum_{s^\tau|s^t} q^t_\tau(s^\tau)[c_{it}(s^t) - y_{it}(s^t)]$$

This will help us to show that $\tilde{c}_{it}(s^t) = c_{it}(s^t)$. In the initial period, $\tilde{a}_{i0}(s_0) = 0$ for all i, thus, substituting the above equation in the sequential markets budget constraint gives:

$$\tilde{c}_{i0}(s_0) + \sum_{t=1}^{\infty}\sum_{s^t} q^0_t(s^t)[c_{it}(s^t) - y_{it}(s^t)] = y_{i0}(s_0) + 0$$

The Arrow-Debreu budget constraint holds with equality which then implies that $c_{i0}(s_0) = \tilde{c}_{i0}(s_0)$ as there is no initial distribution of wealth, the households must rely on its own endowment stream to finance consumption.

Moving to time $t>0$ and histories s^t, replace $\tilde{a}_{it}(s^t)$ in the sequential budget constraint in terms of the net claims $\Upsilon_{it}(s^t)$, we can re-write the value of the asset portfolio as:

$$\sum_{s_{t+1}} \tilde{a}_{it+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) = \Upsilon_{it}(s^t) - [c_{it}(s^t) - y_{it}(s^t)]$$

Now, given that the period-by-period sequential budget constraint holds with equality, this implies that $c_{it}(s^t) = \tilde{c}_{it}(s^t)$ for all i, t, s^t.

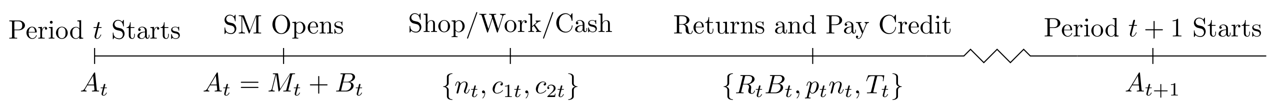
Note that in this case, the household will not increase current consumption by reducing the asset portfolio because of the natural debt limits. The natural debt limit is the weakest possible debt limit for every time period t and history s^t which makes it feasible for the consumer to repay his debt in every possible state and $c_{it}(s^t)$ remains non-negative. Thus, at any period t, the household will not increase consumption because it would not keep him on the optimal path from period $t+1$ onwards for all possible histories.

8 Monetary Economies

In this section, we'd like to present some models of money and address certain properties and concepts within that environment. In particular, we will focus on topics including the opportunity cost of money, neutrality (and super neutrality) of money, the Fisher equation, the Friedman rule and models with both determinate and indeterminate nominal price levels. All models in this section rest upon the underlying *shopper-worker* assumption as well as the *cash-in-advance* (CIA) constraint, which together determine the sequence of actions taken and the frictions faced by agents in these money models.

8.1 Cash-Credit Model

For this model, it is extremely important to understand the timing of the events that take place from period-to-period to understand how the economy evolves and the agent makes decisions. Use the graph timeline below to help understand the sequence of events within a given period.



We assume that a household enters each period with nominal assets (or nominal wealth) A_t and decides how to split this up between current-period money M_t and bonds B_t where the bond promises to pay R_t dollars in interest at the end of the period. So, at the very beginning of the period, the household decides over the division

$$A_t = M_t + B_t. \tag{1}$$

In this economy, there are two consumption goods and a nominal price level p_t, measuring the *dollars* per consumption good in period t (or some other currency index). Note that this price is different than the typical arrow price we have discussed in non-monetary economies. The two goods are the cash good (c_{1t}) and the credit good (c_{2t}). for the cash good, we have the cash-in-advance constraint (CIA) that imposes

$$p_t c_{1t} \leq M_t \tag{2}$$

Thus, you can only consume as much of the cash good as your current money holdings will allow. You can interpret this commodity as a *pay up front* commodity. On the other hand, the credit good can be placed on some type of short-term credit balance to be repaid at the end of the period, or at the very beginning of the following period.[28] The household has preferences over the two goods and leisure, given by

$$\sum_t \beta^t u(c_{1t}, c_{2t}, 1 - n_t) \tag{3}$$

where the utility function is strictly increasing in leisure $(1 - n_t)$ and the consumption goods along with the usual conditions to guarantee an interior solution. Agents wish to optimize (3) with respect to the CIA constraint (2)

[28] In reality, cash goods can be thought of as checking accounts, as well. Examples of cash goods would be groceries, restaurant meals, paying for fuel at a gas station, drinks at a bar, etc. On the other hand, credit goods could be objects such as utility services, where you receive the service and our expected to pay within a month or year's time.

and a sequential budget constraint

$$A_{t+1} + p_t c_{1t} + p_t c_{2t} = M_t + p_t n_t + T_t + R_t B_t \tag{4}$$

and initial conditions $M_0 + B_0 = A_0$. Because next-period assets will be split between next-period cash and bonds, we may equivalently view the budget constraint as

$$M_{t+1} + B_{t+1} + p_t c_{1t} + p_t c_{2t} = M_t + p_t n_t + T_t + R_t B_t \tag{4*}$$

depending on whichever you find more intuitive or convenient. I find (4) most intuitive, as it says that whatever is not spent on cash or credit goods (i.e. the period's net revenues) is simply rolled into next period's asset holdings A_{t+1}.

So, the consumers current period money holdings, labor returns, lump sum tax revenues and end-of-period return on bonds act as within-period revenues, divided between next-period asset holdings and current-period consumption of the cash and credit good. Let's refer back to the timeline and formally run through the sequence of actions. The agent enters period t with some quantity of wealth/assets A_t. The agent immediately goes to the securities markets and divides the assets into cash and bonds. Next, the agent works (some quantity n_t) and buys both c_{1t} and c_{2t}. This is the *shopper-worker* dichotomy. The agent's purchase of c_{1t} is restricted by the CIA and financed solely through the previously chosen money holdings M_t. Lastly, at the end of the period, the agent receives income returns from labor services, bond investment and government transfers. He uses this to pay for the credit good, and pockets the remainder as A_{t+1} for next period.

We further impose a no-ponzi condition

$$\frac{B_t}{p_t} = b_t \geq -\overline{B} \tag{5}$$

for some scalar $B \gg 0$, along with a government budget constraint

$$R_t B_t^s + T_t = B_{t+1}^s + (M_{t+1}^s - M_t^s). \tag{GBC}$$

The LHS can be interpreted as the government expenditures where the government pays its receipts on bonds and transfers to households while the RHS can be viewed as revenues through debt-financing and money creation, respectively.[29] In addition, throughout these notes, I will use capital letters M and B to denote the nominal quantity of money and bonds, whereas lower case letters m and b will refer to their *real* value, after being divided by the same period price level.

Definition 8.1. *A competitive equilibrium is defined by the allocations* $\{(c_{1t}, c_{2t}, l_t)\}_{t=0}^{\infty}$*, quantities* $\{(M_t, B_t)\}_{t=0}^{\infty}$ *and prices* $\{(p_t, R_t)\}_{t=0}^{\infty}$ *such that i) households maximize (3) subject to constraints (1), (2), (4*), (5) and the initial asset conditions and ii) markets clear:*

1. $c_{1t} + c_{2t} = n_t \quad \forall t$
2. $R_t B_t^s + T_t = B_{t+1}^s + (M_{t+1}^s - M_t^s) \quad \forall t$
3. $M_t^s = M_t \quad \wedge \quad B_t^s = B_t \quad \forall t.$

[29] $T_t > 0$ is a lump-sum transfer and $T_t < 0$ denotes a lump-sum tax.

We assume that the sum of cash and credit goods is equal to total amount of labor employed by the agent in each period; thus, there is a 1-to-1 production rate and conversion rate between the two goods. Apply Lagrange multipliers of λ_t and μ_t to the household budget constraint and CIA, respectively. While the agent chooses holdings of money and bonds for the current period, we will perform the maximization with respect to choices of m_{t+1} and c_{t+1} without a loss of generality.[30] This yields the following system of first-order conditions

$$[c_{1t}]: \beta^t u_{1t} = p_t(\lambda_t + \mu_t) \tag{6}$$

$$[c_{2t}]: \beta^t u_{2t} = \lambda_t p_t \tag{7}$$

$$[n_t]: \beta^t u_{nt} = \lambda_t p_t \tag{8}$$

$$[B_t]: \lambda_{t-1} = \lambda_t R_t \tag{9}$$

$$[M_{t+1}]: \lambda_t = \lambda_{t+1} + \mu_{t+1}. \tag{10}$$

Let's do some simplifying. Plug condition (10) into (6) to get $\beta^t u_{1t} = p_t \lambda_{t-1}$. Then divide this by equation (7) to obtain

$$\begin{aligned} \frac{u_{1t}}{u_{2t}} &= \frac{p_t \lambda_{t-1}}{\lambda_t} \\ &= R_t \end{aligned} \tag{11}$$

where the last line comes from employing equation (9). Loosely speaking, condition (11) requires that the marginal rate of substitution between consumption of the cash good and the credit good is equal to the gross rate of interest for bonds. Assuming strictly increasing, concavity and Inada conditions on the utility function, as the return on bonds increases (holding all else constant) the LHS must adjust by decreasing the ratio of consumption between cash over credit goods. In other words, higher interest rates increase the opportunity cost of holding money balances (which have lower return).[31] Next, divide equation (7) by equation (8) to obtain

$$\frac{u_{2t}}{u_{nt}} = 1. \tag{12}$$

Lastly, let's attain an inter-temporal condition with respect to the cash good. For this, let's divide the t condition

[30] Check for yourself that it doesn't matter whether you maximize with respect to $\{M_t, B_t\}$ or $\{M_{t+1}, B_{t+1}\}$.

[31] You may be wondering where the *labor wage* is within the budget constraint. In this model, given the production function $n_t = c_{1t} + c_{2t}$, the wage rate turns out to be $w_t = 1$. To see why this is, consider the firm's problem:

$$\max_{n_t} \quad p_t[n_t - w_t n_t]$$
$$s.t. \quad c_{1t} + c_{2t} \leq n_t.$$

Taking the first-order derivative, we arrive at the condition:

$$p_t[1 - w_t] = 0 \Rightarrow w_t = 1.$$

(6) by its $t+1$ counterpart

$$\begin{aligned}\frac{\beta^t u_{1t}}{\beta^{t+1}u_{1t+1}} =& \frac{p_t(\lambda_t+\mu_t)}{p_{t+1}(\lambda_{t+1}+\mu_{t+1})}\\ =&\frac{1}{\Pi_{t+1}}\frac{\lambda_{t-1}}{\lambda_t}\\ =&\frac{R_t}{\Pi_{t+1}}\\ \Rightarrow \frac{u_{1t}}{\beta u_{1t+1}} =& \frac{R_t}{\Pi_{t+1}} \end{aligned} \tag{13}$$

Here, we use the formula Π_{t+1} to measure the inflation (or gross rate of change) in the price level p_{t+1}/p_t, from period t to period $t+1$. Doing the same for the credit good condition, we observe

$$\frac{u_{2t}}{\beta u_{2t+1}} = \frac{R_{t+1}}{\Pi_{t+1}} \tag{13*}$$

Equation (13*) is a famous term (the Fisher Equation) in disguise. The LHS of the equation is the ratio of marginal utilities for current and next-period consumption of the cash good, measured in date t present value. Given that these terms are relatively close to 1 and after some re-arranging, we arrive at

$$\begin{aligned} R_{t+1} =&(\frac{u_{2t}}{\beta u_{2t+1}})\Pi_{t+1}\\ \Rightarrow 1+r_{t+1} =&(1+\hat{r}_{t+1})(1+\pi_{t+1}) \end{aligned} \tag{14}$$

where the lower-case variables represent *net* values from the gross terms. This identity is approximately represented by $r_t = \hat{r}_t + \pi_t$. Generally, this is interpreted as saying that the nominal return on bonds is equal to the sum of the real return and the inflation of the price level. Bonds are held over the period in expectation of the total return R_t. The opportunity cost of holding these bonds is the ratio of marginal utility received from consuming the credit good. Thus, if bonds are to be attractive investments, they must be at least as high as this cost. Further, during the time of holding bonds, additional money may be created; thus, as money enters the economy, it increases the price level p_t to p_{t+1}. Therefore $3 today does not buy as much tomorrow and the investor must be compensated for this inflation. In total, the nominal return on a bond must be enough to compensate for the opportunity cost of consumption plus the decrease in purchasing power from inflation.

8.1.1 Opportunity Cost of Money

Now, let us rearrange the budget constraint to make some observations about the use of money within this model. First, recall that the Arrow-Debreu price q_t represents the price of a unit of consumption at time t, measured in terms of period 0 consumption. We can express this price as

$$q_t = \frac{1}{R_t}\frac{p_t}{p_{t-1}}\frac{1}{R_{t-1}}\frac{p_{t-1}}{p_{t-2}}\cdots = \frac{p_t}{\Pi_{j=0}^t R_j},$$

or more simply as $q_t = \frac{1}{R_t}\frac{p_t}{p_{t-1}}q_{t-1}$. What does this expression say? The value of time t consumption measured in units of the time 0 good is equal to the nominal price level at time t divided by the product of government debt interest rates, up to time t.

Now, let us multiply the household budget constraint by $\frac{q_t}{p_t}$ to get

$$\frac{q_t M_{t+1}}{p_t} + \frac{q_t B_{t+1}}{p_t} = \frac{q_t M_t}{p_t} - q_t c_{1t} + q_t n_t - q_t c_{2t} + \frac{q_t T_t}{p_t} + \frac{q_t R_t B_t}{p_t} \tag{14}$$

Next, add up the household budget constraints from time 0 to time T:

$$\sum_{t=0}^{T} \frac{q_t}{p_t} M_{t+1} + \sum_{t=0}^{T} \frac{q_t}{p_t} B_{t+1} \leq \sum_{t=0}^{T} \frac{q_t}{p_t} M_t - \sum_{t=0}^{T} q_t[c_{1t} + c_{2t} - n_t] + \sum_{t=0}^{T} \frac{q_t}{p_t} R_t B_t + \sum_{t=0}^{T} \frac{q_t}{p_t} T_t$$

which can be further compacted to

$$\sum_{t=0}^{T-1} [\frac{q_t}{p_t} - \frac{q_{t+1}}{p_{t+1}}] M_{t+1} + \sum_{t=0}^{T-1} [\frac{q_t}{p_t} - \frac{q_{t+1}}{p_{t+1}} R_{t+1}] B_{t+1} + \frac{q_T}{p_T} M_{T+1} + \frac{q_T}{p_T} B_{T+1} \leq \frac{q_0}{p_0} M_0 - \sum_{t=0}^{T} q_t[c_{1t} + c_{2t} - n_t - \frac{T_t}{p_t}] + \frac{q_0}{p_0} R_0 B_0. \tag{15}$$

Next, plug the identity $q_{t+1} = \frac{1}{R_{t+1}} \frac{p_{t+1}}{p_t} q_t$ into the q_{t+1} term for next-period bonds in equation (15) to get a series of cancellations, leading to

$$\sum_{t=0}^{T-1} [\frac{q_t}{p_t} - \frac{q_{t+1}}{p_{t+1}}] M_{t+1} + \frac{q_T}{p_T} M_{T+1} + \frac{q_T}{p_T} B_{T+1} \leq \frac{q_0}{p_0} [M_0 + R_0 B_0] - \sum_{t=0}^{T} q_t[c_{1t} + c_{2t} - n_t - \frac{T_t}{p_t}]. \tag{16}$$

We now impose a transversality-type assumption:

$$\lim_{n \to \infty} \quad q_{t+1} \frac{M_{t+2}}{p_{t+1}} + q_{t+1} \frac{B_{t+2}}{p_{t+1}} = 0. \tag{TVC}$$

Why are we justified in making this assumption? If this were not true, and instead the quantity $q_{t+1}[\frac{M_{t+2}}{p_{t+1}} + \frac{B_{t+2}}{p_{t+1}}] > 0$ in the limit. Then, this quantity can be marginally decreased and consumption of goods c_{1t} and c_{2t} can be marginally increased on the RHS of (16). Given strictly increasing preferences, this would necessarily increase expected lifetime utility; thus, we impose this condition at the equilibrium. In taking the limit of (16), we observe

$$\sum_{t=0}^{\infty} [\frac{q_t}{p_t} - \frac{q_{t+1}}{p_{t+1}}] M_{t+1} = \frac{q_0}{p_0} [M_0 + R_0 B_0] - \sum_{t=0}^{\infty} q_t[c_{1t} + c_{2t} - n_t - \frac{T_t}{p_t}]$$

$$\Rightarrow \sum_{t=0}^{\infty} \frac{q_{t+1}}{p_{t+1}} [R_{t+1} - 1] M_{t+1} + \sum_{t=0}^{\infty} q_t[c_{1t} + c_{2t} - n_t + \frac{T_t}{p_t}] = \frac{q_0}{p_0} [M_0 + R_0 B_0] \tag{17}$$

where we normalize the date-0 Arrow-Debreu price to $q_0 = 1$ and we observe inequality because this condition holds at the equilibrium. Getting to the last line requires $M_{t+1}[\frac{q_t}{p_t} - \frac{q_{t+1}}{p_{t+1}}] = \frac{q_{t+1}}{p_{t+1}} M_{t+1} [\frac{p_{t+1}}{p_t} \frac{q_t}{q_{t+1}} - 1] = \frac{q_{t+1}}{p_{t+1}} M_{t+1} [R_{t+1} - 1]$ through playing with the equation for q_t. Lastly, a little manipulation and use of the initial conditions, we can isolate A_0 on the right hand side:

$$\begin{aligned} \frac{q_0}{p_0} [M_0 + R_0 B_0] &= \frac{q_0}{p_0} M_0 + \frac{q_0}{p_0} R_0 B_0 \\ &= \frac{q_0}{p_0} M_0 + \frac{q_0}{p_0} R_0 (A_0 - M_0) \\ &= -\frac{q_0}{p_0} (R_0 - 1) M_0 + \frac{q_0}{p_0} R_0 A_0 \end{aligned} \tag{18}$$

which can be plugged into (17) for the final representation

$$\sum_{t=0}^{\infty} \frac{q_t}{p_t} [R_t - 1] M_t + \sum_{t=0}^{\infty} q_t[c_{1t} + c_{2t} - n_t + \frac{T_t}{p_t}] = \frac{q_0}{p_0} R_0 A_0. \tag{19}$$

We will refer to equation (19) as the *consolidated budget constraint.* Observe that the RHS is an expression for the real value of initial wealth for the representative agent. This should somewhat remind you of the implementability constraint in Ramsey problems. Thus, initial wealth is divided between the time 0 value of consumption less work/taxes and a term containing a measure of real money balances through time. If $R_t > 1$, the money component has positive value in the equation and then the consumption/labor component is reduced, necessarily leading to a reduction in lifetime utility. In this sense, nominal interest rates in excess of 1 represent the opportunity cost of money. More on this later.

Let us now consider two different stationary settings in which the money supply, quantity of bonds and lump-sum transfers are fixed in some sense. First, consider the basic case in which $M_t = \bar{M}, B_t = \bar{B}$ and $T_t = \bar{T} \quad \forall t$. So, government money creation, bond sales and transfers are fixed quantities for every period. This further will imply that the price level and interest rate are fixed at $\bar{p}$ and $\bar{R}$, respectively. Going back to the inter-temporal equation (13*), we simply have

$$\bar{R} = \frac{1}{\beta} \tag{20}$$

as quantity of consumption is constant, as well. This is equating the nominal interest rate to the inverse of the discount factor in equilibrium. If we instead let the money supply grow at some constant rate $\theta > 1$ such that $M_{t+1} = (1+\theta)M_t$, then the price level would adjust accordingly as $p_{t+1} = (1+\theta)p_t$ and we would observe

$$R = \frac{1}{\beta}\Pi = \frac{1}{\beta}(1+\pi) = \frac{1}{\beta}(1+\theta). \tag{21}$$

This is simply a unique case of the Fisher equation. The term representing the real interest rate on bonds is simplified due to a constant level of consumption through time. Thus, the real rate of return for the bonds (or its opportunity cost of investing) is simply the inverse of the discount factor.

8.1.2 Money Neutrality and Superneutrality

Let's now discuss neutrality. In this model, if we were to take two economies with objects (c_{1t}, c_{2t}, n_t) with money stock M_t and a separate economy with objects $(\hat{c}_{1t}, \hat{c}_{2t}, \hat{n}_t)$ and money stock αM_t where $\alpha > 0$, all else is the same and these objects are equilibria, then the allocations for consumption and labor/leisure are identical. Look at our model in a steady state with constant money stock, as characterized by the FOC in (20). A one-time increase to the stock of money leaves the first-order condition unaffected and thus doesn't change the quantity of real variable. We would call this model economy *neutral.*

Definition 8.2. *An economy is money neutral if the equilibrium allocations are invariant to a one-time change in the stock of money.*

Let's say you are living on an island with one other inhabitant, Jeff. You and Jeff are pretty sophisticated and enact an enforceable currency system with the 10 one-dollar bills you found. In this economy, the price of coconuts eventually becomes \$1. What happens when ten more dollars wash up on shore one day? The money supply doubles, the price level doubles, coconuts become \$2 and your equilibrium allocation does not change at all. This is a crude example of money neutrality. The price level changes but the real quantities (the things that actually matter for utility) do not change. Now, let's move to another common property that some monetary economies exhibit: super neutrality.

Definition 8.3. *An economy is super neutral if all real variables are invariant/independent with respect to changes in the steady state long-run growth of the money supply.*

For our cash-credit model, we introduced two different types of stationary equilibria: one with a constant stock of money supply and the other with a growing supply, at some rate θ. When this is the case, we saw that the equilibrium gross interest rate on government bonds was set to $R = \frac{\Pi}{\beta}$. If the rate of money growth and hence inflation was altered to $\theta' > \Pi$, we would see an increase in the equilibrium interest rate to R'. What implications follow from this? With higher R, the opportunity cost of money (or the *money wedge*) is increased, leading to a decrease in lifetime utility and an affect on the real quantities in the economy. Thus, our model economy does not exhibit super neutrality.

8.1.3 Optimal Quantity of Money and the Friedman Rule

Now that we've presented the structure of the model and some of the interactions that are taking place, we can address the question of optimality. When the CIA constraint binds, there can be a certain cost to holding money. This cost is reflected in the fact that agents could be investing their assets A_t in bonds, which earn some return $R \geq 1$. Let's first look at the social planner's approach to our economy:

$$max \quad \sum \beta^t u(c_{1t}, c_{2t}, 1 - n_t)$$
$$s.t. \quad c_{1t} + c_{2t} = n_t$$

which leads to first-order conditions:

$$[c_{1t}] : \beta^t u_{1t} - \lambda_t = 0$$
$$[c_{2t}] : \beta^t u_{2t} - \lambda_t = 0$$
$$[n_t] : \beta^t u_{nt} - \lambda_t = 0$$

which implies that in the optimum we have

$$\frac{u_{1t}}{u_{2t}} = 1. \tag{22}$$

Thus, for a competitive equilibrium to attain the social optimum, we must equate equations (11) and (22) by setting nominal gross interest $R_t = 1$. Let's hark back to the Fisher equation. Under this setup, it is advised to set the net interest rate to zero (i.e. $r = 0$). The equation becomes

$$\pi = -\hat{r}.$$

with the real net interest rate in the economy set equal to the negative of the net inflation rate. Thus, the optimal policy is to pursue deflation in the economy, slowly shrinking the money supply. This insight is attributed to Milton Friedman and is called the Friedman rule. Does this sound right? One line to support this result is equation (19). Having a nominal gross rate of 1 would cancel the first component of the LHS of the equation. In effect, this would maximize the amount of initial wealth that could be spent on real quantities. In addition, recall that there is an opportunity cost to holding money, and the only reason for holding the asset is that it is required to consume the cash good. What is this opportunity cost? The return R_t that one could get from a bond. When $R_t > 1$, the CIA constraint will bind and the consumer will choose a sub-optimal level of consumption of the cash good, due to the constraint. This constraint binds because the nominal quantity of money simply has a return of 1. Thus, as an investment vehicle, it is an inferior product. By shrinking the money supply, the *real* value of a household's money is increasing at a rate equivalent to that of the bond return. Thus, under the Friedman rule, households will consume the optimal quantity of goods c_{1t} and c_{2t} as if there were no CIA constraint.

How is this policy implemented? The government must steadily decrease the money supply, so it may implement this policy through lump-sum taxes on household money stocks at the end of the period.

8.2 Stochastic Cash-Credit Model

Now, we may augment the model to exist in a stochastic environment in which output is some random variable of an underlying shock. In particular, assume output $y(s)$ is exogenously given with s as a random variable in a finite state space and following a Markov process with a stationary transition function $\pi(s'|s)$. For some initial simplification, we will take the labor/leisure decision and bonds out of the current model. Thus, consumers simply seek to maximize

$$\sum_t \sum_{s^t} u(c_{1t}(s^t), c_{2t}(s^t)). \tag{1}$$

The randomness comes in the forms of shocks to the consumer's endowment, as well as the growth rate of the money supply. For the government, the growth rate of money is defined by the function $g(s^t)$ which is a gross rate measure of money supply growth. Let's further characterize the economy's aggregate stock of money as $\overline{M}_t$ at time t. The government budget constraint is simply to transfer the money stock to households through a lump-sum transfer scheme. Thus, we observe the following aggregate law of motion for the money supply

$$\overline{M}_t(s^t) = g(s^t)\overline{M}_{t-1} = \overline{M}_{t-1}(s^{t-1}) + T_t(s^t) \tag{2}$$

in period t. In this model, we will consider stationary equilibrium; thus, without loss, we will drop the time subscripts on the relevant variables. The CIA constraint is given by

$$p(s)c_1(s) \leq M(s) \tag{3}$$

and the consumer faces the period budget constraint

$$M'(s') = M(s) + p(s)[y(s) - c_1(s) - c_2(s)] + T'(s'). \tag{4}$$

A couple of things to pay attention to: The agent no longer enters the next period with assets A_{t+1} which may be divided into current-period money and bonds. Now, the agent simply goes from period to period with a certain quantity of money, determined by their previous period consumption and the stochastic transfer from the government. Thus, in period t an agent implicitly chooses money stock M_{t+1} by determining how much to consume of c_{1t} and c_{2t}. The remainder of money after that plus the end-of-period transfer from the government $T_{t+1}(s^{t+1})$ determines the cash available for period $t+1$. Thus, the agent chooses *pre-transfer* cash holdings, given expectation of the transfers T_{t+1}.[32]

Before moving on, let's divide both the CIA and budget constraint by the current period aggregate stock of money $\overline{M}_t$. This leads to

$$\hat{p}(s)c_1(s) \leq \hat{M}(s) \tag{3*}$$

where *hatted* variables are for when objects are divided by the aggregate level of that period's money stock. Further, we have

$$\hat{M}'(s')g(s') = \hat{M}(s) + \hat{p}(s)[y(s) - c_1(s) - c_2(s)] + g(s') - 1. \tag{4*}$$

For the LHS, we use the operation $\frac{M'}{\overline{M}} = \frac{M'}{\overline{M}'}\frac{\overline{M}'}{\overline{M}} = \hat{M}'g(s')$ and given that $T' = \overline{M}' - \overline{M}$, we get the expression $g(s') - 1$ after dividing by the current period money supply.

We can now recast this problem as a functional equation for the consumer. Given the current state, as defined by money holdings and current-period shock, the agent must make decisions with respect to current consumption of c_1 and c_2, as well as next-period pre-transfer cash holdings M'. We may write

$$v(M,s) = \max_{\{c_1(s),c_2(s)\}} \{u(c_1(s), c_2(s)) + \beta \sum_{s'} \pi(s'|s)v(M', s')\} \tag{5}$$

subject to

$$\begin{aligned} \hat{p}(s)c_1(s) \leq& \hat{M}(s) && \text{(CIA)} \\ \hat{M}'(s')g(s') =& \hat{M}(s) - \hat{p}(s)[y(s) - c_1(s) - c_2(s)] + g(s') - 1. && \text{(BC)} \\ & c_{1t}(s), c_{2t}(s) \geq 0 \end{aligned}$$

In the Lucas-Stokey paper, they specify the choice of pre-transfer balances as x_3. In this setup, $x_3 = \hat{M}(s) + \hat{p}(s)[y(s) - c_1(s) - c_2(s)]$, which leads to next-period money balances of $\hat{M}'(s') = \frac{x_3+g(s')-1}{g(s')}$ where the *per capita* money holdings are normalized to one in each period (i.e. $\hat{M}(s) = 1$). Under strong enough assumptions for the utility function and correspondence defining the feasible set of choices, we can prove the existence of such a value function v which implies the existence of policy functions for the choice variables.

Definition 8.4. *In this economy, a recursive competitive equilibrium is defined by a value function v, policy functions for $\{\hat{c}_1, \hat{c}_2\}$,, a pricing system $p(s)$ such that*

i. v is a solution to the functional equation (5),

ii. policy functions g_{c1} and g_{c2} solve the functional equation, and

iii. markets clear:

[32] This notation is a little confusing because we have T_{t+1} in the period t budget constraint. This may be rewritten differently to achieve the same desired result. The only point being made is that the agent is not certain of how much cash she will have on hand in the following period.

a. $c_1(s) + c_2(s) = y(s) \quad \forall s \in S$

b. $\overline{M}'(s') = \overline{M}(s) + T(s') \quad \forall s, s' \in S$

c. $\hat{M}(s) = 1 \quad \forall s \in S.$

For the market-clearing condition $(iii)(c)$, we require that the aggregate supply of money, given any shock, is equal to the aggregate post-transfer money holdings for the representative agent. Recall, $\hat{M}$ represents *per capita* holdings; thus, the condition requires that $M(s) = \overline{M}(s)$.

This model may easily be extended to include a one-period bond. In particular, we may include B' which is a bond, promising to pay 1 unit of the consumption good in the following period. Further, this bond is associated with a price $q'(s')$. Now, once again, the agent will enter the current period with $A(s)$ in nominal assets,

$$M(s) + q(s)B(s) = A(s)$$

with the same CIA constraint and a slightly augmented budget constraint

$$A'(s') = M(s) + p(s)[y(s) - c_1(s) - c_2(s)] + T'(s') + B(s).$$

Under this setup, there are two primary results from the Lucas-Stokey paper. Result 1: Given that the shock s is *iid*, the price of the one period bond $q(s)$ is independent of s. Result 2: Given that s is *iid* and $y(s) = \overline{y}$ for all s, then $p(s)$ is independent of s, implying that $c_1(s)$ and $c_2(s)$ are independent of s, as well. Result 1 claims that if shocks to worker income and the money growth rate are independent and identically distributed, then the pricing of claims to next-period units of consumption no longer depend upon those stochastic processes. Further, if the shocks are *iid* and only affect the money growth rate, the price level (and hence inflation) are independent of the stochastic process for s. In addition, consumption decisions are made independent of the shock to the money supply in every period.

8.3 Price Indeterminacy

Often we care whether or not the nominal variables in the model have been uniquely pinned down. Consider this first, formal definition of an indeterminate variable.

Definition 8.5. *Let $\{x_t\}_{t=0}^{\infty}$ be some equilibrium sequence, consisting of a price system and allocations for the households and firms. This sequence is **indeterminate** if for some neighborhood N of $\{x_t\}$, every sequence $\{\hat{x}_t\} \in N$ is also an equilibrium.*

In this sense, an economy has multiple equilibria if both $\{x_t\}$ and $\{\hat{x}_t\}$ are equilibrium; thus, indeterminacy implies multiplicity of equilibria. Specifically, when we are considering the *price system*, we may solve for all of the *real* variables in the model but fail to find one particular price that satisfies the equilibrium conditions.

8.3.1 Classic Monetary Model

Consider the following example, based upon the model and results presented in Chapter 2 of (Galí, 2015). In what follows, let capital letters denote nominal variables and lowercase letters for their log transform. A representative agent wishes to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t[Z_t])$$

where C_t is consumption, N_t is labor at time t and Z_t is a stochastic shock. This maximization is subject to the budget constraint

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t \tag{1}$$

where P_t is the price of the consumption good, W_t is the nominal wage and B_t is the price of a one-period, risk-free bond. Also, given that the representative firm has a production function $Y_t = A_t N_t^{1-\alpha}$, this leads to the first order conditions

$$[C_t]: \quad -\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} \tag{2}$$

$$[N_t]: \quad Q_t = \beta^t \{\frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}}\} \tag{3}$$

$$[N_t^f]: \quad \frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha} \tag{4}$$

along with a transversality condition on the real value of one-period bonds.

Now, let us consider the utility function

$$U(C_t, N_t) = [logC_t - \frac{N_t^{1+\phi}}{1+\phi}] Z_t$$

and let us assume the following exogenous process on both productivity and stochastic shock parameters:

$$a_t = \rho_a a_{t-1} + \epsilon_t^a \tag{5}$$

$$z_t = \rho_z z_{t-1} + \epsilon_t^z \tag{6}$$

where the coefficients are elements of $[0,1]$ and $a_t = logA_t$ and $z_t = logZ_t$. Given the specific utility function above, first-order condition (2) becomes $W_t P_t = C_t^{\sigma} N_t^{\phi}$, which is

$$w_t - p_t = \sigma c_t + \phi n_t \tag{7}$$

in log-transformed terms. In addition, and using a Taylor-series expansion around the steady state, the log-transformed version of first-order condition (3) is

$$c_t = E_t[c_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - \rho) + \frac{1}{\sigma}(1-\rho_z) z_t \tag{8}$$

where $i_t = -logQ_t$ is the nominal interest rate, $\pi_{t+1} = p_{t+1} - p_t$ is the inflation at date t and $\rho = -log\beta$ is the discount rate.[33] In addition, the log-transformed firm first-order condition (4) becomes

$$w_t - p_t = log(1-\alpha) + a_t - \alpha n_t. \tag{9}$$

[33] The Euler equation can be rewritten as $1 = \frac{\beta}{Q_t} E_t[\frac{U_{c,t+1}}{U_{c,t}} \frac{Z_{t+1}}{Z_t} \frac{P_t}{P_{t+1}}]$. This implies

$$1 = E_t[exp(i_t - \rho - \sigma \delta c_{t+1} + \delta z_{t+1} - \pi_{t+1})], \tag{*}$$

where we now utilize the log-transformed variables. A steady state with constant productivity growth γ and constant inflation π (in which $\delta c_t = 0$) is characterized by $i = \rho + \sigma\gamma + \pi$. Thus, a first-order Taylor series expansion of the exponent above, around the constant

Lastly, let us introduce/postulate a money demand equation for real balances.

$$m_t - p_t = c_t - \eta i_t \tag{10}$$

where $\eta \geq 0$. For now, this demand equation is more or less coming out of thin air, but we will use for purposes of this example.

Equate equations (7) and (9) and sub in y_t for c_t to get an equation in terms of n_t and y_t. Then, given the log-transformed production function $y_t = a_t + (1-\alpha)n_t$, we can solve for y_t and n_t:

$$n_t = \frac{1-\sigma}{\sigma(1-\alpha)+\phi+\alpha}a_t + \frac{log(1-\alpha)}{\sigma(1-\alpha)+\phi+\alpha} \tag{11}$$

$$y_t = \frac{1+\phi}{\sigma(1-\alpha)+\phi+\alpha}a_t + \frac{(1-\alpha)log(1-\alpha)}{\sigma(1-\alpha)+\phi+\alpha}. \tag{12}$$

Next, making use of the identity $i_t = r_t + E_t[\pi_{t+1}]$, known as the Fisher equation, we can solve for the real interest rate r_t by utilizing (8), market-clearing conditions and the assumed processes (5) and (6):

$$r_t = \rho + (1-\rho_z)z_t - \frac{\sigma(1-\rho_a)(1+\phi)}{\sigma(1-\alpha)+\phi+\alpha}a_t. \tag{13}$$

Given our results for c_t and y_t in (12) and n_t in (11), we can derive the real wage $w_t - p_t$ from equations (7) or (9) by making the proper substitutions. Thus, the real variables and the real prices are determined, absent of knowledge of monetary policy, which is dictated by (10).

To examine a case in which the price level is not determined, consider a monetary policy where the nominal interest rate is exogenous and of the form

$$i_t = i + v_t \tag{14}$$

where v_t follows the process

$$v_t = \rho_v v_{t-1} + \epsilon_t^v,$$

where we will refer to the v_t's as monetary policy shocks. From examining (13), if there is no growth (in the form of shocks z_t and a_t), then the steady state real interest rate will be $r = \rho$. Thus, in a perfect foresight, steady state equilibrium, we will observe

$$i = \rho + \pi.$$

steady state, is

$$\begin{aligned} exp(i_t - \rho - \sigma\delta c_{t+1} + \delta z_{t+1} - \pi_{t+1}) &= 1 + (i_t - i) - \sigma(\Delta c_{t+1} - \gamma) + \Delta z_{t+1} - (\pi_{t+1} - \pi) \\ &= 1 + i_t - \sigma\Delta c_{t+1} - \pi_{t+1} + \Delta z_{t+1} - \rho \end{aligned}$$

where we make use of the steady state identity above for i. Thus, to equate the LHS of (*), we must have $1 = exp(1)$, which implies

$$c_t = E_t[c_{t+1}] - \frac{1}{\sigma}[i_t - E_t[\pi_{t+1}] - \rho - (1-\rho_z)z_t]$$

which is the approximate Euler equation, where I made use of

$$E_t[\Delta z_{t+1}] = E_t[\rho_z z_t + \epsilon_t^z - z_t] = (\rho_z - 1)z_t.$$

Thus, consider

$$\begin{aligned} E_t[\pi_{t+1}] =& i_t - r_t \\ =& i + v_t - r_t \\ =& \pi + v_t - \hat{r}_t \end{aligned}$$

where $\hat{r}_t = r_t - \rho$ and the third equality makes use of the perfect foresight, steady state equilibrium quality for i. Thus, expected inflation is uniquely pinned down, but now consider actual inflation $\pi_t = p_t - p_{t-1}$. There does not exist another condition that can uniquely solve for this value. Therefore, any equation satisfying

$$\pi_t = \pi + v_{t-1} - \hat{r}_{t-1} + u_t$$

where u_t is a shock, satisfying $E_{t-1}[u_t] = 0$ constitutes an equilibrium. This can be rewritten for the period t price level as

$$p_t = p_{t-1} + \pi + v_{t-1} - \hat{r}_{t-1} + u_t.$$

Thus, we have *price level indeterminacy* in this classic monetary model when the monetary policy corresponds to the money demand equation (10) and a stochastic, exogenous process for the nominal interest rate. When the process on monetary policy or the interest rate rule, we can have situations in which the price level is uniquely pinned down. Refer to the Galí text for these.

8.3.2 Woodford Model

For another example of price indeterminacy, I will briefly review the deterministic model from Michael Woodford.[34] Consider the same discounted utility maximization problem, where we assume that u is strictly concave in its credit and cash good arguments and define

$$\hat{c}_1 = \underset{c_1}{argmax}\ u(c_1, y - c_1)$$

where we assume $0 < \hat{c}_1 < y$, showing that the maximizing argument of the utility function is interior and less than the deterministic endowment y. The representative agent is subject to the following constraints:

$$p_t c_{1t} \leq M_t \tag{1}$$

$$M_t + E[r_{t+1} B_{t+1}] \leq A_t + T_t \tag{2}$$

$$A_{t+1} = M_t + p_t(y - c_{1t} - c_{2t}) + B_{t+1} \tag{3}$$

as well as non-negativity constraints on consumption, a no-ponzi condition on asset holdings and initial assets A_0 is given. Equation (1) is the classic CIA constraint for the cash good. Equation (2) defines the securities market restrictions that the agent faces every period. In particular, Given incoming assets A_t and a government transfer T_t, the agent allocates this between current-period nominal money and purchases of next-period bonds, promising to pay one unit (measured in some abstract unit of account). The government faces a similar budget constraint

$$M_t + E[r_{t+1} B_{t+1}] = M_{t-1} + B_t + T_t. \tag{GBC}$$

Without spending time to re-derive, the paper shows that if a competitive equilibrium exists, it is necessary and sufficient to observe:

$$c_{1t} = \frac{M_t}{p_t} \tag{4}$$

$$c_{2t} = y - c_{1t} \tag{5}$$

$$\frac{u_2(m_t)}{p_t} = \beta E[\frac{u_1(m_{t+1})}{p_{t+1}}] \tag{6}$$

$$u_2(t) r_{t+1} = \beta \frac{p_t}{p_{t+1}} u_1(t+1) \tag{7}$$

Essentially, we find that at bond interest rates $R_t > 1$, the CIA constraint binds, pinning down cash good consumption as equal to the real money balances being held. Further, the household resource constraint holds within each period. Using lowercase notation for real money balances, equation (6) provides an inter-temporal and cross-goods equation for marginal rates of substitution, and equation (7) is the more familiar inter-temporal constraint for the cash good. After some substitutions, the equilibrium can be sufficiently characterized by

$$r_{t+1} = \beta \frac{p_t}{p_{t+1}} \frac{u_1(m_{t+1})}{u_1 m_t}. \tag{8}$$

Now, as a simple case, let's consider an constant money growth fiscal policy, where the money stock grows according to

$$M_t = M_0 g^t \tag{9}$$

[34] *Monetary Policy and Price Level Determinacy in a CIA Economy* from *Economic Theory* 4 (1994).

where g is some constant, greater than 1. Further, the government implements this policy through lump sump transfers, using the operation

$$T_t = (g-1)M_{t-1}. \tag{10}$$

Assuming a stationary equilibrium, the Euler equation (6) simplifies to

$$u_2(m^*) = \frac{\beta}{g}u_1(m^*). \tag{11}$$

where m^* is real money balances, satisfying $m^* = \frac{M}{p}$ from the CIA constraint. Now, a couple results from the paper.

Proposition 8.1. *With additively separable preferences (i.e.* $u(c_1, c_2) = u(c_1)+u(c_2)$*) and exogenous money supply growth* g*, if i)* $g < \beta$*, there does not exist a steady state equilibrium, ii) if* $g = \beta$*, there exists a continuum of steady state equilibria and iii) if* $g > \beta$*, steady state equilibria are locally isolated, if they do exist.*

In a following section, titled *self-fulfilling inflations*, we can consider the following assumption

$$\lim_{c\to 0} cu_1(c) = 0. \tag{13}$$

Proposition 8.2. *Suppose that peferences and the rate of money growth* g *satisfy assumptions (12), (13) and the usual assumption on the utility function* u*. Then there exists a continuum of perfect foresight equilibria in each of which real money balances approach zero asymptotically, and also an infinite number of sunspot equilibria in which real money balances approach zero asymptotically with positive probability.*[35]

The paper goes on to characterize many different conditions that are sufficient and/or necessary for unique equilibria determination under both the constant money growth regime and an *interest rate peg* regime.

[35] Refer to the paper for definitions of perfect foresight equilibrium and sunspot equilibrium.

9 Sticky/Stagnant Price Models

There exists evidence in data regarding slow adjustment of aggregate price levels (macroeconomic evidence) and in significant price and wage rigidities in markets (microeconomic evidence). If our objective is to develop economic models whose results are close to the data, so that we can further understand how this economic variables work, we need to include some stickiness to prices in our models. We will work with the basic New Keynesian model, in which monopolistic firms can adjust their prices in each period with a constant probability $1-\theta$. This is based on the staggered price-setting model of Calvo (1983).

9.1 About the staggered price-setting model

Imagine there is a fairy in the economy, which we refer to as Calvo fairy. This being appears at the beginning of each t, chooses fraction $(1-\theta)$ of firms randomly and taps them on the shoulder, giving them the permission to change their prices. This occurs every period.

A more realistic story is that there are small costs of changing prices in the economy, referred to as menu costs. Assuming a unit mass of firms in each period, a fraction θ of firms find it too costly to adjust their prices to unanticipated shocks in the economy.

For a firm that sets its price in period t, there is a $(1-\theta)$ chance of resetting price next period. With probability θ, the firm has to keep its price fixed in period $t+1$. So, the chance of a one-period rigidity is $\theta(1-\theta)$. Therefore the expected length of price rigidity is

$$\sum_{k=0}^{\infty} \theta^k (1-\theta) k$$

which is a geometric power series converging to $1/(1-\theta)$.

9.2 Environment

There are three types of agents in the economy:

1. Households

   Representative agent infinitely lived whose preferences over consumption of different varieties of goods, labor, and money are represented by the following utility function in expected form:

   $$E_0 \sum_{t=0}^{\infty} \beta^t [U(C_t, N_t) + V(M_t/P_t)]$$

   where C_t is a consumption index for each differentiated good i. We will assume there is a exogenous continuum of such goods in the economy, of unit mass.

2. Firms

   There is a unit mass continuum of monopoly firms (they choose their own price) that produce each differentiated good i. In each period, fraction $1-\theta$ of the firms have the chance to adjust their prices, while fraction θ has to keep their price fixed and meet the demand in the market.

3. Government

   The government prints money and supplies it to the household, emits and pays bonds traded with the consumer, and gives transfer to the consumer. We assume will assume stochastic money growth that will generate an ad-hoc money demand equation.

9.3 Households

Problem of the representative household is:

$$\max_{C_t(i,s^t),B_{t+1}(s^t),N_t(s^t)} E_0 \sum_{t=0}^{\infty} \beta^t [U(C_t(s^t), N_t(s^t)) + V(M_t(s^t)/P_t(s^t))]$$

$$\text{s.t.} \;\; C_t(s^t) = \Big[\int_0^1 C_t(i,s^t)^{\frac{\epsilon-1}{\epsilon}} di\Big]^{\frac{\epsilon}{\epsilon-1}}$$

$$\int_0^1 P_t(i,s^t)C_t(i,s^t)di \; + \; M_t(s^t) \; + \; Q_t(s^t)B_{t+1}(s_t,s^t) \leq$$

$$B_t(s^{t-1}) \; + \; W_t(s^t)N_t(s^t) \; + \; T_t(s^t) \; + \; M_{t-1}(s^{t-1}) \; + \; \Pi_t(s^t)$$

$$\lim_{T\to\infty} E_t(B_t) \geq 0$$

The last condition rules out the possibility of Ponzi schemes in the economy. For notation purposes, we are omitting the history dependence of the variable, but they should be taken into account.

To solve the household problem we need two different steps: (1) to decide the fraction of wealth household wants to allocate to consumption, and then (2) divide this fraction between the different varieties of consumption goods. We will begin with the latter; given an amount W of wealth dedicated to consumption and price of each differentiated good, how the household divides W between varieties. Since we have separable utility with money, we can write the problem as follows:

$$\max_{C(i)} U(C,N)$$

$$\text{s.t.} \;\; C = \Big[\int_0^1 C(i)^{\frac{\epsilon-1}{\epsilon}} di\Big]^{\frac{\epsilon}{\epsilon-1}}$$

$$\int_0^1 P(i)C(i)di = W$$

The first order conditions for variety i:

$$U_c(C,N)C(i)^{\frac{-1}{\epsilon}} \Big[\int_0^1 C(k)^{\frac{\epsilon-1}{\epsilon}} dk\Big]^{\frac{1}{\epsilon-1}} = \lambda P(i) \;\; \forall i \in (0,1)$$

where λ is the lagrange multiplier on the budget constraint.

Dividing the first order conditions for varieties i and j we have:

$$\frac{C(i)^{-1/\epsilon}}{C(j)^{-1/\epsilon}} = \frac{P(i)}{P(j)}$$

$$C(i) = \left[\frac{P(j)}{P(i)}\right]^{\epsilon} C(j)$$

Plugging back the last equation to the definition of C,

$$C = \left[\int_0^1 \left[\left[\frac{P(j)}{P(i)} \right]^{\epsilon} C(j) \right]^{\frac{\epsilon-1}{\epsilon}} di \right]^{\frac{\epsilon}{\epsilon-1}}$$

$$C = \frac{P(j)^{\epsilon}}{\left[\int_0^1 P(i)^{1-\epsilon} di \right]^{\frac{\epsilon}{1-\epsilon}}} C(j)$$

$$C(j) = \left[\frac{P(j)}{P} \right]^{-\epsilon} C$$

where $P \equiv \left[\int_0^1 P(i)^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}}$ is the aggregate price index.

Additionally, we can rewrite the budget constraint as:

$$W = \int_0^1 P(i)C(i)di = \int_0^1 P(i)\Big[\frac{P(i)}{P}\Big]^{-\epsilon} C di = PC$$

using the definition for price index in the last equality.

Regarding the decision between consumption and labor, the consumer's problem can be written as,

$$\max_{C_t(is^t), B_{t+1}(s^t), N_t(s^t)} E_0 \sum_{t=0}^{\infty} \beta^t [U(C_t(s^t), N_t(s^t)) + V(M_t(s^t)/P_t(s^t))]$$

$$P_t(s^t)C_t(s^t) \ + \ M_t(s^t) \ + \ Q_t(s^t)B_{t+1}(s^t) \leq B_t(s^{t-1}) \ + \ W_t(s^t)N_t(s^t) \ + \ T_t(s^t) \ + \ M_{t-1}(s^{t-1}) \ + \ \Pi_t(s^t)$$

$$\lim_{T \to \infty} E_t(B_t) \geq 0$$

The first order conditions are defined as follows:

$$\beta^t \pi_t(s^t) U_C(s^t) = P_t(s^t) \lambda_t(s^t)$$

$$\beta^t \pi_t(s^t) U_N(s^t) = -W_t(s^t) \lambda_t(s^t)$$

$$\beta^t \pi_t(s^t) V_M(s^t)(1/P_t(s^t)) = \lambda_t(s^t) - \sum_{s_{t+1}|s^t} \lambda_{t+1}(s^{t+1})$$

$$Q_t(s^t)\lambda_t(s^t) = \sum_{s_{t+1}|s^t} \lambda_{t+1}(s_{t+1})$$

$$V_M(s^t) = U_C(s^t)(1 - Q_t(s^t))$$

The optimality conditions derived from the first order conditions:

$$\frac{U_N(t)}{U_C(t)} = -\frac{W_t}{P_t}$$

$$Q_t = \beta E_t \Big[\frac{U_C(t+1)P_t}{U_C(t)P_{t+1}} \Big]$$

Now, lets assume that the utility function takes the following separable form,

$$U(C, N) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\phi}}{1+\phi} + \frac{(M/P)^{1+\alpha}}{1+\alpha}$$

Then the optimality conditions can be written as

$$1 = E_t\Big(\beta \frac{1}{Q_t}\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\Big)$$

$$(M_t/P_t)^{\alpha} = C^{-\sigma}(1-Q_t)$$

$$\frac{N_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t}$$

The Euler Equation can be written in natural log terms as:

$$1 = E_t(\exp(\log\beta + i_t - \sigma\Delta c_{t+1} - \pi_{t+1}))$$

where $i_t \simeq log(1/Q_t)$, $\Delta c_{t+1} = log(C_{t+1}) - log(C_t) = c_{t+1} - c_t$, $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$, and lower case letters represent the logarithm of the variables. (Note that $log(1+i_t) \equiv log(1/Q_t) \simeq i_t$.)

On a balanced growth path, assuming consumption grows at a constant rate γ, we have

$$1 = \exp(\log\beta + i - \sigma\gamma - \pi)$$

$$i = -\log\beta + \sigma\gamma + \pi$$

A first order Taylor expansion around the balanced growth path yields:

$$\exp(\log\beta + i_t - \sigma\Delta c_{t+1} - \pi_{t+1}) \simeq 1 + \log\beta + i_t - \sigma\Delta c_{t+1} - \pi_{t+1}$$

Substituting the last result in the Euler Equation gives

$$c_t = E_t(c_{t+1}) - \frac{1}{\sigma}(\log\beta + i_t - E_t(\pi_{t+1}))$$

The optimality condition between labor and consumption can also be expressed in logarithmic terms,

$$w_t - p_t = \sigma c_t + \phi n_t$$

9.4 Monopoly firms

The continuum of monopoly firms produce each good i with the **same technology**:

$$Y_t(i) = A_t N_t(i)$$

Firm i faces a demand function from the consumers,

$$C_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\epsilon} C_t$$

and takes price index P_t and aggregate consumption C_t as given. Firms change their prices in Calvo Fairy style. They discount future profits with $q_{t,t+k}(s^{t+k})\pi(s^{t+k}|s^t)$ which is the value of one unit in t to be delivered in $t+k$ given history s^{t+k}.

The problem faced by the firm choosing in t to adjust its price is as follows.

$$\max_{P_t(i)} \sum_{k=0}^{\infty} \theta^k E_t\Big(q_{t,t+k}(P_t(i)Y_{t+k|t}(i) - \Psi_{t+k}(Y_{t+k|t}(i)))\Big)$$

$$\text{s.t.} \;\; Y_{t+k|t}(i) = \left[\frac{P_t(i)}{P_{t+k}}\right]^{-\epsilon} C_{t+k}$$

where Ψ is the cost function.

The value of $q_{t,t+k}(s^{t+k})\pi(s^{t+k}|s^t)$ corresponds to the price of a bond bought in period t (with history s^t) and received in period $t+k$ given history s^{t+k}. To obtain te value let's consider the following endowment economy consumer's problem:

$$\max_{c_t(s^t), b_{t+k}(s^{t+k}|s^t)} E_0 \sum_t \beta^t u(c_t(s^t))$$

$$\text{s.t.} \;\; p_t(s^t)c_t(s^t) + q_{t,t+k}(s^{t+k})\pi(s^{t+k}|s^t)b_{t+k}(s^{t+k}|s^t) \leq p_t(s^t)y_t(s^t) + b_t(s^t)$$

Then the first order conditions of the problem are:

$$\beta^t \pi_t(s^t) u_c(s^t) = \lambda_t(s^t) p_t(s^t)$$

$$\lambda_t(s^t) q_{t,t+k}(s^{t+k})\pi(s^{t+k}|s^t) = \lambda_{t+k}(s^{t+k})$$

Then the optimality condition for the price of the bonds:

$$q_{t,t+k}(s^{t+k}) = \beta^k \frac{u_c(s^{t+k})}{u_c(s^t)} \frac{p_{t+k}(s^{t+k})}{p_t(s^t)}$$

Returning to the monopoly firm's problem, we can substitute for the demand function, and rewrite the problem as

$$\max_{P_t(i)} \sum_{k=0}^{\infty} \theta^k E_t\Big(q_{t,t+k}(P_t(i)P_t(i)^{-\epsilon}P_{t+k}^{\epsilon}C_{t+k} - \Psi_{t+k}(P_t(i)^{-\epsilon}P_{t+k}^{\epsilon}C_{t+k}))\Big)$$

The first order conditions for this problem with respect to $P_t(i)$ (and substituting back for $Y_{t+k|k}(i)$),

$$\sum_{k=0}^{\infty} \theta^k E_t\left(q_{t,t+k}\Big((1-\epsilon)Y_{t+k|k}(i) + \epsilon P_t(i)^{-1} Y_{t+k|t}(i) \frac{\partial \Psi_{t+k}(Y_{t+k|t})}{\partial Y_{t+k|t}(i)}\Big)\right) = 0$$

If we multiply both sides of the equality by $P_t(i)/(1-\epsilon)$,

$$\sum_{k=0}^{\infty} \theta^k E_t\Big(q_{t,t+k} Y_{t+k|k}(i)\Big(P_t(i) - \frac{\epsilon}{\epsilon-1}\psi_{t+k|k}(i)\Big)\Big) = 0$$

where $\psi_{t+k|t}(i) \equiv \dfrac{\partial \Psi_{t+k|t}(i)}{\partial Y_{t+k|t}(i)}$ is the nominal marginal cost of the firm on period $t+k$.

If technology is the same for every firm, then $\psi_{t+k|t}(i) = \psi_{t+k|t}(j)$ which implies that $P_t(i) = P_t(j) = P_t^*$.

Note that if $\theta = 0$, with frictionless optimal pricing, the condition is equal to

$$P_t^* = \frac{\epsilon}{\epsilon - 1}\psi_{t+k|t}$$

which means that the price of the monopolistic firm is marked up above the marginal cost by a constant that depends on the elasticity of demand. Denote this markup $\mathcal{M} \equiv \frac{\epsilon}{\epsilon-1}$.

Dividing this equation by P_{t-1} (price index at time $t-1$) and substituting for inflation $\Pi_{t-1,t+k} \equiv \frac{P_{t+k}}{P_{t-1}}$ we can rewrite this condition as:

$$\sum_{k=0}^{\infty} \theta^k E_t \Bigg(q_{t,t+k} Y_{t+k|t} \Big(\frac{P_t^*}{P_{t-1}} - \mathcal{M} \frac{\psi_{t+k|t}}{P_{t+k}} \frac{P_{t+k}}{P_{t-1}} \Big) \Bigg) = 0$$

$$\sum_{k=0}^{\infty} \theta^k E_t \Bigg(q_{t,t+k} Y_{t+k|t} \Big(\frac{P_t^*}{P_{t-1}} - \mathcal{M} \; MC_{t+k|t} \; \Pi_{t-1,t+k} \Big) \Bigg) = 0$$

where $MC_{t+k|t}$ is the **real** marginal cost in period $t+k$ of a firm that sets its price at t.

Consider the **steady state** of this economy. In this case, since consumption and prices remain constant ($P_t^* = P_t = P_{t+1}$), inflation is equal to one, and $Q_{t,t+k} = \beta^k$.

Then the first order condition implies,

$$MC_{t+k|t} = \frac{1}{\mathcal{M}}$$

We can log-linearize the firm's first order condition around the zero-inflation steady state as:

$$p_t^* - p_{t-1} = (1-\beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k E_t (mc_{t+k|t} + m + p_{t+k} - p_{t-1})$$

Note: To log-linearize an equation of the form $f(X_t, Y_t) = g(Z_t)$ around the steady state, use the following formula:

$$g'(Z) Z z_t \simeq f_1(X,Y) X x_t + f_2(X,Y) Y y_t$$

where X, Y, and Z are the steady-state values of variables X_t, Y_t, Z_t and the small letters denote natural logarithms.

9.5 Aggregate Price Level Dynamics

Let $S_t \subset [0,1]$ be the subset of firms that did not adjust their prices at t. The aggregate price index in period t is given by

$$P_t^{1-\epsilon} = \int_{S_t} P_{t-1}(i)^{1-\epsilon} di + \int_{S_t^c} P_t(i)^{1-\epsilon} di$$

For the fraction $(1-\theta)$ firms that are allowed to change its price in t, we know they will choose P_t^*.

$$\int_{S_t^c} P_t(i)^{1-\epsilon} di = (P_t^*)^{1-\epsilon} \int_{S_t^c} di = (1-\theta)(P_t^*)^{1-\epsilon}$$

For the fraction θ of firms that couldn't change price in t we have

$$\int_{S_t} P_{t-1}(i)^{1-\epsilon} di = \theta \int_0^1 P_{t-1}(i)^{1-\epsilon} di$$

and applying the definition for the price index in $t-1$, P_{t-1},

$$\theta \int_0^1 P_{t-1}(i)^{1-\epsilon} di = \theta P_{t-1}^{1-\epsilon}$$

Hence, the aggregate price level at t can be written as

$$P_t^{1-\epsilon} = \theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}$$

Dividing the last equation by P_{t-1} we get

$$\Big(\frac{P_t}{P_{t-1}}\Big)^{1-\epsilon} = \theta + (1-\theta)\Big(\frac{P_t^*}{P_{t-1}}\Big)^{1-\epsilon}$$

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta)\Big(\frac{P_t^*}{P_{t-1}}\Big)^{1-\epsilon}$$

In the **zero-inflation steady state** inflation equals one, and prices remain constant, $P_t^* = P_t = P_{t-1}$. Then if we log-linearize around the steady state we get

$$(1-\theta)(p_t^* - p_{t-1}) = \pi_t$$

9.6 Equilibrium

Definition 9.1. *A* ***monetary equilibrium with sticky prices*** *is given by household allocation* $\{C_t(s^t), C_t(i, s^t), N_t(s^t), M_t(s^t), B_{t+1}(s^t)\}$*, firm's allocation* $\{Y_{t+k|t}\}$*, policy allocation* $\{M_t^g(s^t), B_{t+1}^g(s^t), T_t(s^t), q_{t,t+k}\}$*, and prices* $\{P_t(i, s^t), P_t(s^t), Q_t(s^t), W_t(s^t)\}$ *s.t.*

(i) household's allocation solve their problem given prices and policy, (ii) the monopoly firm's chose price given the demand of their product, (iii) government satisfies it's budget constraint, $M_t(s^t) + M_{t-1}(s^{t-1}) + Q_t(s^t)B_{t+1}(s^t) = B_t(s^{t-1}) + T_t(s^t)$*, and (iv) labor and consumption good markets clears.*

Solving for the equilibrium. Given technology for firms $Y_t(i) = A_t N_t(i)$, the cost function is

$$\Psi_t(i) = W_t N_t(i) = W_t \frac{Y_t(i)}{A_t}$$

Then the **real** marginal cost in period $t+k$ of a firm that sets its price in t is

$$MC_{t+k|t} = \frac{W_{t+k}}{A_{t+k}P_{t+k}}$$

which is the same for all firms (no matter at what time they adjust their price), and, hence, equal to MC_{t+k}. Applying logarithm to the last equation

$$mc_{t+k} = w_{t+k} - p_{t+k} - a_{t+k}$$

Substituting this result into the firm's log-linearized first order condition around the zero-inflation steady state,

$$p_t^* - p_{t-1} = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(mc_{t+k} + m) + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(p_{t+k} - p_{t-1})$$

We can expand this equation as

$$
\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(mc_{t+k}+m) \\
&\quad + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k(p_{t+k}-p_{t+k-1}+p_{t+k-1}-p_{t+k-2}+p_{t+k-2}+...+p_t-p_{t-1}) \\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(mc_{t+k}+m) \\
&\quad + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(\pi_{t+k}+\pi_{t+k-1}+...+\pi_t)
\end{aligned}
$$

Which is equal to

$$
\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(mc_{t+k}+m) \\
&\quad + (1-\beta\theta)(\pi_t+\beta\theta\pi_t+(\beta\theta)^2\pi_t)+...) \\
&\quad + (1-\beta\theta)(\beta\theta\pi_{t+1}+(\beta\theta)^2\pi_{t+1}+...)+... \\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(mc_{t+k}+m) \\
&\quad + (1-\beta\theta)\Big[\frac{\pi_t}{1-\beta\theta}+\beta\theta\frac{\pi_{t+1}}{1-\beta\theta}+...\Big] \\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(mc_{t+k}+m) + \sum_{k=0}^{\infty}(\beta\theta)^k E_t(\pi_{t+k})
\end{aligned}
$$

Re-writing this equation for $t+1$, by the Law of Iterated Expectations, we have

$$E_t(p_{t+1}^* - p_t) = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^{k+1} E_t(mc_{t+k+1}+m) + \sum_{k=0}^{\infty}(\beta\theta)^{k+1} E_t(\pi_{t+k+1})$$

$$E_t(p_{t+1}^* - p_t) = (1-\beta\theta)\sum_{k=1}^{\infty}(\beta\theta)^{k} E_t(mc_{t+k}+m) + \sum_{k=1}^{\infty}(\beta\theta)^{k} E_t(\pi_{t+k})$$

Then we can write it as

$$p_t^* - p_{t-1} = \beta\theta E_t(p_{t+1}^* - p_t) + (1-\beta\theta)(mc_t+m) + \pi_t$$

And substituting for the log-linearized equations in the steady state, $(1-\theta)(p_t^*-p_{t-1}) = \pi_t$

$$\frac{\pi_t}{1-\theta} = \beta\theta E_t\Big(\frac{\pi_{t+1}}{1-\theta}\Big) + (1-\beta\theta)(mc_t+m) + \pi_t$$

$$\pi_t = \beta E_t(\pi_{t+1}) + \frac{(1-\theta)(1-\beta\theta)}{\theta}(mc_t+m)$$

Now, looking into the **labor market**, we know

$$Y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\epsilon} Y_t$$
$$Y_t(i) = A_t N_t(i)$$

and the labor-market clearing condition states

$$N_t = \int_0^1 N_t(i) di$$
$$N_t = \int_0^1 \frac{Y_t(i)}{A_t} di$$
$$N_t = \frac{Y_t}{A_t} \int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\epsilon} di$$

Taking the logarithm of the last equation

$$n_t = y_t - a_t + d_t$$

where d_t is a measure of price dispersion across firms. Around the zero-inflation steady state, $d_t = 0$, and we can write this equation as:

$$n_t = y_t - a_t$$

Substituting from the good's market clearing condition into the household's optimality condition for labor supply:

$$w_t - p_t = \sigma y_t + \phi n_t$$
$$w_t - p_t = \sigma y_t + \phi(y_t - a_t)$$

And considering the value for the mc_t,

$$\begin{aligned} mc_t &= w_t - p_t - a_t \\ &= \sigma y_t + \phi(y_t - a_t) - a_t \\ &= (\sigma + \phi) y_t - (1 + \phi) a_t \end{aligned}$$

In the case of flexible prices, we have shown that $mc_t^n = -m$ and in the linearized equation $mc_t^n = (\sigma + \phi) y_t^n - (1 + \phi) a_t$. Then we can show

$$mc_t + m = (\sigma + \phi)(y_t - y_t^n) \equiv (\sigma + \phi)\tilde{y_t}$$

where $\tilde{y_t} = y_t - y_t^n$ are the deviations from the flexible pricing model.

Substituting in our last equation for inflation, we obtain the **New-Keynesian Phillips Curve**,

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa \tilde{y_t}$$

where $\kappa = \frac{1}{\theta}(1 - \theta)(1 - \beta\theta)(\sigma + \phi)$.

Now looking at the log-linearized form of the Euler Equation,

$$c_t = E_t(c_{t+1}) - \frac{1}{\sigma}[log\beta + i_t - E_t(\pi_{t+1})]$$

Substituting $c_t = y_t$ and adding y_t^n on both sides we obtain the **Dynamic IS Equation**,

$$\tilde{y_t} = E_t(\tilde{y}_{t+1}) - \frac{1}{\sigma}[i_t - E_t(\pi_{t+1}) + \log\beta - \sigma E_t(\Delta y_{t+1}^n)]$$

where $\tilde{y_t} = y_t - y_t^n$.

The New-Keynesian Phillips Curve and the Dynamic IS Equation, together with a policy equation that determines the dynamics of nominal interest rate, i_t, characterize the equilibrium of the economy.

The process governing i_t depends on how monetary policy is conducted, which in our case is through ad-hoc money demand equation (this in contrast to classical models where monetary policy is neutral). Here, monetary policy is a key determinant of real changes in the economy. For instance, an equation usually noted to describe how monetary authority reacts to the inflation rate and output gap is the **Taylor rule**,

$$i_t = -\log\beta + \alpha_\pi \pi_t + \alpha_y \tilde{y}_t + \nu_t$$

where α_π and α_y are non-negative and ν_t is an exogenous component with zero mean.

10 Sustainable Equilibrium

Often within monetary or fiscal policy models, we assume the government has some credible commitment technology. This means that the government will honor in the future whatever it announces it will do today. For example, say the government announces a constant rate of money growth θ for all future time periods and this government is *benevolent*, meaning it seeks to optimize aggregate welfare. Even though a one-time positive shock in the growth rate of money may increase social welfare, the government will not do so because of its previous commitment. Such assumptions add a good deal of simplicity to modeling the environment. While convenient, this assumption is unrealistic, as modern economic history contains many episodes of sovereign debt defaults, inflation crises, failed currency pegs and political backtracking. The key insight here is that government authorities can only commit to an action if that action is aligned with their incentives not just today, but at all future dates and states of nature. In this section, we will review the formal way in which to look at these type of equilibria.

10.1 Chari-Kehoe Model

This subsection reviews the paper and results from V.V. Chari and Pat Kehoe's *Sustainable Plans* from the *Journal of Political Economy* (1990). The environment is a little more complicated than that of the previous subsection, but still obeys the general definition, provided in Definition 10.2. Below is an illustration, demonstrating the timing of events in this model.

As you can see, a single time period is characterized by a morning and night (i.e. AM or PM) which determines the sequence of events that takes place between the representative household and government. For any given time period t, the household gains utility from AM consumption (c_{1t}), PM consumption (c_{2t}) and leisure ($1 - l_t$). Over an infinite horizon, utility is summarized by the function

$$\sum_{t=0}^{\infty} \beta^t u(c_{1t} + c_{2t}, 1 - l_t). \tag{1}$$

In each period, the household starts the day with the endowment ω and determines how to split this between consumption and capital investment:

$$c_{1t} + k_t \leq \omega. \tag{2}$$

This capital earns a rate of interest $R \geq 1$, which is to be realized at night/PM. It is also during the PM stage that the household chooses labor supply, earning a one-to-one return of consumption for labor. The PM is also the time in which the government may impose a labor tax τ and a capital tax δ. Thus, the PM budget constraint is

$$c_{2t} \leq (1 - \delta_t)Rk_t + (1 - \tau_t)l_t. \tag{3}$$

As for the government, it faces a balanced budget constraint for each period and must raise the exogenously determined revenue g; thus, we have

$$g = \delta_t R k_t + \tau_t l_t, \tag{4}$$

and in order to guarantee that the government does not rely solely on taxing capital, we impose the assumption

Assumption 10.1. $g > (R-1)\omega$.

As can be inferred from the timeline above, the sequence of events is as follows: While in the AM, consumers make decisions with respect to current consumption and capital investment. During the transition from AM to PM, the government imposes its current policy, specifying taxes on capital and on labor. Once in the PM, the consumer makes decisions (with respect to the government's policy) as to the amount of labor to employ and consumption. Thus, for any period t, we have a household allocation rule $f_t = (f_{1t}, f_{2t}) = ((c_{1t}, k_t), (l_t, c_{2t}))$ and a government policy $\pi_t = (\tau_t, \delta_t)$. Note: Let's maintain the same understanding from the last subsection that π_t implicitly defines all future period policies; that is, $\pi_t = \{\tau_t, \delta_t, \tau_{t+1}, \delta_{t+1}, ...\}$. We may further compact the household rule and government policy into the sequences $f = \{f_t\}_{t=0}^{\infty}$ and $\pi = \{\pi_t\}_{t=0}^{\infty}$, respectively.

Recall that a Ramsey equilibrium loosely refers to an environment with commitment in which the government chooses (at date 0) the fiscal policy that maximizes welfare. In particular, given knowledge of how households respond to a policy π, the government picks the π that maximizes welfare in a competitive equilibrium setting, and does so with respect to household and government budget constraints.

Definition 10.1. *A **Ramsey equilibrium** is a policy π and household allocation rule f such that*

1. *For all $\hat{\pi}$,*
$$f(\hat{\pi}) = argmax \sum_{t=0}^{\infty} \beta^t u(c_{1t} + c_{2t}, 1 - l_t)$$
subject to the household budget constraint for all t, and

2. *the government chooses a fiscal policy π such that*
$$\pi = argmax \sum_{t=0}^{\infty} \beta^t u(c_{1t}(\pi) + c_{2t}(\pi), 1 - l_t(\pi))$$
subject to the household and government budget constraints for all t.

Thus, given a credible commitment strategy and knowing the household's best response (as characterized by the sequence of functions f), the government maximizes utility in a way to meet the household constraints and its own revenue requirements. Given this equilibrium notion, the paper proves the following proposition.

Proposition 10.1. *The Ramsey outcome (π^r, f^r) has AM allocations $c_{1t}^r = 0$ and $k_t^r = \omega$ and a capital tax rate $\delta_t^r = \frac{R-1}{R}$ for every period.*

While not used in the section 10.1, we'll introduce the notion of a *history*, as it is included in the paper. A history h_{t-1} is simply the set of all past government policies, up until date t; that is, $h_{t-1} = \{\pi_0, \pi_1, ..., \pi_{t-1}\}$. Thus, at the beginning of period t, the household is aware of the history of policies and makes a decision accordingly, such that $f_{1t}(h_{t-1})$. The government's period t policy is also a function of this history. Lastly, given the period policy π_t, the history is updated to h_t and the PM decision$f_{2t}(h_t) = f_{2t}(h_{t-1}, \pi_t)$ is made by the household.

We haven't forgotten about the policy mapping either! Instead of the constant function σ for all t, we consider the sequence of mappings $\sigma = \{\sigma_1, \sigma_2, ...\}$ for this model. By the time we get to period t, the government observes

h_{t-1} and chooses period t policy by $\sigma_t(h_{t-1})$. We have a sustainable equilibrium if $\sigma_t(h_{t-1})$ coincides with all previous specifications of π_t.[36]

Definition 10.2. *A sustainable equilibrium is a pair (π, f) along with the sequence of policy mappings σ that satisfy the following conditions:*

1. *Given the policy π and mapping σ, the continuation of the allocation rule f solves the consumer's problem, during the AM, for every history h_{t-1}, and the continuation of the allocation rule solves the consumer's problem, during the PM, for every history h_t.*
2. *Given an allocation rule f, continuation plan of σ solves the government's problem for every history h_{t-1}.*
3. *Markets clear.*

(1) specifies that households respond optimally to each history and these responses coincide with the initial allocation rules f; (2) states that the government responds optimally to each history in a way that maximizes competitive welfare of the household, satisfies its own budget constraint and coincides with its initial policy plan π.

Theorem 10.1. *In this setting, the autarky equilibrium (σ^a, f^a) is the worst sustainable equilibrium. That is, for any sustainable equilibrium (σ, f), $U(\sigma, f) \geq U(\sigma^a, f^a)$, where U represents lifetime utility of the household.*

Autarky refers to the scenario in which the household chooses to not roll over or invest any of its capital for consumption in the PM stage for any period. Lastly, the main result of the paper.

Theorem 10.2. *An arbitrary pair of realized outcomes (π, f) is a sustainable equilibrium if and only if*

1. *the pair (π, f) is a competitive equilibrium at date 0 and*
2. *for every period t, the following inequality holds:*

$$\sum_{s=t}^{\infty} \beta^{s-t} u(c_{1s} + c_{2s}, l_s) \geq u^d(k_t) + \frac{\beta}{1-\beta} u^a$$

where u^a is utility earned from autarky and u^d is the one-period maximized utility of the household, subject to the PM constraint and the government budget constraint.

Refer to Lemma 2 and the actual theorem in the paper for a more explicit understanding of the notation. At face value, think of u^d as the *one shot* highest utility a household can get in period t while still respecting the relevant constraints. If u^a is autarky utility, then $\frac{\beta}{1-\beta}$ is the present value of an infinite stream of autarky outcomes for the household.

What is the best sustainable outcome we can obtain?

We know the Ramsey outcome will yield the highest utility for the household, but, is it sustainable?

Theorem 10.3. *Folk's Theorem. There exists a discount factor $\beta^* \in (0, 1)$ such that if $\beta \geq \beta^*$ then the Ramsey equilibrium is sustainable.*

[36] Why does the government choose time t policy with the history h_{t-1} and not $\{h_{t-1}, f_{1t}(h_{t-1})\}$? Because it is redundant. The government is aware of the household's response function f and thus h_{t-1} is a sufficient statistic for knowing f_{1t}.

Proof. Let u^R be de utility derived from a Ramsey equilibrium, u^d the highest utility derived in a one shot deviation, and u^a the autarky utility. Then the sustainability constraint is

$$\frac{u^R}{1-\beta} \geq u^d + \frac{\beta u^a}{1-\beta}$$

$$u^R \geq (1-\beta)u^d + \beta u^a$$

By assumption, $u^d \geq u^R \geq u^a$. Let β^* be the discount factor such that the sustainability constraint holds with equality,

$$\beta^* = \frac{u^d - u^R}{u^d - u^a} \geq 0$$

Then for all $\beta \geq \beta^*$ the sustainability constraint holds and the Ramsey equilibrium is sustainable.

10.2 Prelim Questions

Setup: Consider a production economy with a large number of idenical, infinitely lived individuals. There are two goods: labor and consumption. These correspond to a period utility function $U(c_t, l_t)$. There exists a per capita government expenditure g_t, determined by a finite, independent and identical stochastic process. The government can finance this with taxes on labor income and issuing one-period debt. The initial stock of government debt is known and positive. Government debt must be non-negative; that is, the household cannot borrow from the government.

A) Define a competitive equilibrium.

A TDCE is a sequence of household allocation rules $Z^H = \{c_t(g_t), l_t(g_t), b_{t+1}(g_t)\}_{t=0}^{\infty}$, a firm production plan $Z^F = \{l_t^f(g_t)\}_{t=0}^{\infty}$, a government policy $\{g_t, \tau_t(g_t), b_{t+1}^g(g_t)\}_{t=0}^{\infty}$ and prices $\{R_t(g_t), w_t(g_t)\}_{t=0}^{\infty}$ such that the household, taking prices as given, solves

$$\max_{Z^H} \sum_{t=0}^{\infty} \sum_{g_t} \beta^t \pi(g_t) u(c_t(g_t), l_t(g_t))$$
$$s.t. \quad c_t(g_t) + b_{t+1}(g_t) \leq (1 - \tau_t(g_t)) w_t(g_t) l_t(g_t) + R_t(g_t) b_t(g_{t-1})$$
$$s.t. \quad b_0 > 0, \quad b_{t+1}(g_t) \geq 0, \quad c_t(g_t) \geq 0 \;\; \forall t, \forall g_t$$

the firm, taking prices as given, solves

$$\max_{Z^F} l_t^f(g_t) - w_t(g_t) l_t^f(g_t) \quad \forall t, \forall (g_t)$$

the government budget constraint holds:

$$g_t + R_t(g_t) b_t(g_{t-1}) = \tau_t(g_t) w_t(g_t) l_t(g_t) + b_{t+1}(g_t) \quad \forall t, \forall (g_t)$$

and markets clear:

$$c_t(g_t) + g_t = l_t(g_t) \quad \text{and} \quad b_t^g(g_t) = b_t(g_t) \quad \text{and} \quad l_t^f(g_t) = l_t(g_t) \quad \forall t, \forall (g_t).$$

B) Suppose that the government wishes to implement the best competitive equilibrium. Set up the Ramsey problem. With a multiplier of $\lambda_t(g_t)$ on the budget constraint, we have first-order conditions

$$\begin{aligned}[c_t(g_t)] :& \beta^t \pi(g_t) u_c(g_t) = \lambda_t(g_t) \\ [l_t(g_t)] :& \beta^t \pi(g_t) u_l(g_t) = (1-\tau(g_t))\lambda_t(g_t) \\ [b_{t+1}(g_t)] :& -\lambda_t(g_t) + \sum_{g_{t+1}} \lambda_{t+1}(g_{t+1}) R_{t+1}(g_{t+1}) = 0.\end{aligned}$$

the consolidation of first order conditions between consumption and labor leads to

$$\frac{u_l(g_t)}{u_c(g_t)} = (1-\tau_t(g_t))w_t(g_t) = (1-\tau_t(g_t)), \tag{1}$$

given that the firm first order condition provides a wage of 1[37], and

$$1 = \beta \sum_{g_{t+1}} \frac{\pi(g_{t+1})}{\pi(g_t)} \frac{u_c(g_{t+1})}{u_c(g_t)} R_{t+1}(g_{t+1}). \tag{2}$$

The procedure for attaining an implementability constraint proceeds as follows: Multiply the period $t+1$ budget constraint by $\beta \frac{\pi_{t+1}(g_{t+1})}{\pi_t(g_t)} \frac{u_c(g_{t+1})}{u_c(g_t)}$ and sum over all realizations of g_{t+1}. You get

$$\begin{aligned}\sum_{g_{t+1}} \beta \frac{\pi_{t+1}(g_{t+1})}{\pi_t(g_t)} \frac{u_c(g_{t+1})}{u_c(g_t)} [c_{t+1}(g_{t+1}) + b_{t+2}(g_{t+1})] &= \sum_{g_{t+1}} \beta \frac{\pi_{t+1}(g_{t+1})}{\pi_t(g_t)} \frac{u_c(g_{t+1})}{u_c(g_t)} [(1-\tau(g_{t+1})) + R_{t+1}(g_{t+1}) b_{t+1}(g_t)] \\ \Rightarrow \frac{\beta}{u_c(g_t)} E_t[u_c(g_{t+1})c_{t+1}(g_{t+1}) + u_c(g_{t+1}) b_{t+2}(g_{t+1})] &= \frac{\beta}{u_c(g_t)} E_t[u_l(g_{t+1})] + b_{t+1}(g_t) \\ \Rightarrow b_{t+1}(g_t) &= \frac{\beta}{u_c(g_t)} E_t[u_c(g_{t+1})c_{t+1}(g_{t+1}) + u_c(g_{t+1}) b_{t+2}(g_{t+1}) - u_l(g_{t+1})],\end{aligned}$$

which is an expression for $b_{t+1}(g_t)$. Now, use this in the current period budget constraint, which is $c_t(g_t) + b_{t+1}(g_t) = (1-\tau(g_t)) + R_t(g_t) b_t(g_{t-1})$, and get

$$\begin{aligned}c_t(g_t) + \frac{\beta}{u_c(g_t)} E_t[u_c(g_{t+1})c_{t+1}(g_{t+1}) - u_l(g_{t+1}) + u_c(g_{t+1}) b_{t+2}(g_{t+1})] = 1 - \tau(g_t) + R_t(g_t) b_t(g_{t-1}) \\ \Rightarrow \frac{1}{u_c(g_t)} E_t[u_c(g_t)c_t(g_t) + \beta u_c(g_{t+1}) c_{t+1}(g_{t+1}) - u_l(g_t) - \beta u_l(g_{t+1})] + \beta E_t[\frac{u_c(g_{t+1})}{u_c(g_t)} b_{t+2}(g_{t+1})] = R_t(g_t) b_t(g_{t-1}).\end{aligned}$$

As can be seen, a pattern will emerge, given the repeated substitution of the next-period bond equation. Thus, assuming that

$$\lim_{T\to\infty} \beta E_t[\frac{u_c(g_{t+T})}{u_c(g_t)} b_{t+T}(g_{t+T-1})] = 0$$

and setting $t = 0$, we get the implementability constraint

$$\sum_{t=0} \beta^t E_0[u_c(g_t)c_t(g_t) - u_l(g_t)] = u_c(g_0) R_0(g_0) b_0,$$

or

$$\sum_t \sum_{g_t} \beta^t \pi(g_t)[u_c(g_t)c_t(g_t) - u_l(g_t) l_t(g_t)] = u_c(g_0) R_0(g_0) b_0 \tag{3}$$

where we assume $\pi(g_0) = 1$ and $l_t(g_t) = 1\ \forall t$ and $\forall g_t$.

[37] Firm's maximize $l_t^f(g_t) - w_t(g_t) l_t^f(g_t)$, where the derivative with respect to $l_t(g_t)$ gives the desired result.

For the planner, with commitment, the Ramsey problem is

$$\max \sum_{t=0}^{\infty} \sum_{g_t} \beta^t \pi(g_t) u(c_t(g_t), l_t(g_t))$$
$$s.t. \quad c_t(g_t) + g_t = l_t(g_t)$$
$$s.t. \quad \sum_t \sum_{g_t} \beta^t \frac{\pi(g_t)}{u_c(g_0)} [u_c(g_t)c_t(g_t) - u_l(g_t)l_t(g_t)] = R_0(g_0)b_0$$

C) Suppose now that the government lacks commitment. Assume policies and allocations can depend on the entire history of policies. Define a sustainable equilibrium

When the government lacks commitment, it is able to renege on promises or future plans that have been made to the household. Within the economic environment of this problem, the only inter-temporal instrument the government uses is bonds. Thus, we will allow for the government to default at a rate $\delta_t(g_t) \in [0, 1]$ on promised bond payments $R_t(g_t)b_t(g_{t-1})$. Notice, given that δ is in the unit interval, we can have partial defaults. Period t repayments are a function of bonds sold in period $t-1$, leading to the inter-temporal aspect of the commitment problem. Within a given period, define a government policy as $\psi_t = (\tau_t(g_t), \delta_t(g_t), (b_{t+1}(g_t))_{g_{t+1}})$ and the household allocation rule as $f_t = (c_t(g_t), (b_{t+1}(g_t))_{g_{t+1}}, l_t(g_t))$. Note: g_t does not appear in the government policy because it is a stochastic and exogenous variable that the government does not control. Now, a definition for sustainable equilibrium.

Definition 10.3. *Let $f = \{f_t\}_{t=0}^{\infty}$ be a sequence of household allocation rules for each time period and let $\psi = \{\psi_t\}_{t=0}^{\infty}$ be a sequence of government policy rules with a corresponding sequence of policy mappings $\sigma = \{\sigma_t\}_{t=0}^{\infty}$ such that $\sigma_t : h_{t-1} \to \psi_t$ where the history h_t is defined as $h_t = (h_{t-1}, \sigma_t(h_{t-1}))$.*[38] *A sustainable equilibrium is the pair (σ, f) such that* ***Added for all time periods in the conditions.***

1. *Given σ, for every time t and $\forall h_t$*

$$\{f_s\}_{s=t}^{\infty} \in argmax \sum_{s=t}^{\infty} \sum_{g_s} \beta^{s-t} \pi(g_s) u(c_s(g_s), l_s(g_s))$$
$$s.t. \quad c_s(g_s) + \sum_{g_{s+1}} b_{s+1}(g_{s+1}|g_s) \leq (1 - \tau_s(g_s))w_s(g_s)l_s(g_s) + \delta_s(g_s)R_s(g_s)b_s(g_{s-1})$$

2. *Given f and h_{t-1}, for every time t,*

$$\{\sigma_s\}_{s=t}^{\infty} \in argmax \sum_{s=t}^{\infty} \sum_{g_s} \beta^{s-t} \pi(g_s) u(c_s(g_s), l_s(g_s))$$
$$s.t. \quad g_s + \delta_s(g_s)R_s(g_s)b_s(g_{s-1}) = \tau_s(g_s)w_s(g_s)l_s(g_s) + \sum_{g_{s+1}} b_{s+1}(g_{s+1}|g_s,)$$

3. *Markets clear*

[38] Implicitly, we are assuming that within any time period, the government makes an action first, followed by a household response.

where future allocations and utilities are induced by the σ and f.[39] [40]

D) Without commitment, what is the best sustainable equilibrium if g_0 is positive and $g_t = 0$ for all $t \geq 1$?

Refer to part E for a generalized proof.

E) Without commitment, what is the best sustainable equilibrium if $g_t = g > 0$ for $t = 1, 2, ..., T$ and $g_t = 0$ for all $t \geq T + 1$?

For the remainder of the question, we will assume that sequential equilibria are characterized by trigger mechanisms, or *revert-to* strategies. This is in line with Chari, Kehoe (1990) and Chari, Kehoe (1993). Here, assume that if the government deviates from its announced policy in period t, then there is autarky for periods $t + 1$ and onward. That is, there is no bond market and the government can only finance expenditures through its labor tax. Lastly, note that we have moved into a deterministic environment.

Claim 10.1. *If $g_t = 0 \quad \forall t \geq \tau$ for some τ, then any sustainable plan must default at τ.*

Proof. Given that the government does not need to finance expenditures after period $\tau - 1$, the government budget constraint becomes: $\delta_t R_t b_t \leq \tau_t l_t + b_{t+1}$. In particular, for the first period τ, we have

$$\delta_\tau R_\tau b_\tau \leq \tau_\tau l_\tau$$

with the government objective of maximizing

$$\sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t, l_t)$$

subject to $c_t + b_{t+1} \leq (1 - \tau_t) l_t + \delta_t R_t b_t \quad \forall t \geq \tau$.[41] If the government defaults in period τ, the household receives welfare $U_d + \frac{\beta}{1-\beta} V$ whereas if the government does not default, the household receives $U_{ND} + \frac{\beta}{1-\beta} V$. Thus, the government's decision is with respect to which period τ utility is higher. Consider the household's feasibility set $(c, l) \in \Gamma(\tau, w)$ such that

$$\Gamma(\tau, w) = \{(c, l) : c \geq 0, l \in [0, 1], c = (1 - \tau) l + w\}$$

where w corresponds to a bond repayment. Given that the government must finance period τ bond repayments with a distortionary tax $\tau l = w$, we note that $\Gamma(\tau, w) \subseteq \Gamma(0, 0)$ for any $\tau, w > 0$. Thus, from a planner's perspective,

[39] When we say that future histories are induced by a pair (σ, f), we are stating that for any date t, the government policy (along with the household allocation rule) maps h_{t-1} into h_t into h_{t+1} and so forth, such that the future household utility can be computed

$$\sum_{s \geq t}^{\infty} \beta^{s-t} U(c_s(h_{s-1}))$$

as a function of the histories. Essentially, knowing how the household and government optimize and respond to one another, we can forecast the future histories of the game, which determine future utility levels.

[40] In the second item, there is no consumer budget constraint included because the response functions, which the government is taking as given, already optimize with respect to this constraint.

[41] This proof assumes that the government does not choose to roll over debt with $b_{\tau+1}$. Given a no-ponzi condition and episolon argument, this proof will follow through for the more general case in which the government is allowed to roll over some debt.

facing resource constraint $c = l$, the maximum is attained at $\frac{u_l}{u_c} = 1$. Further, when faced with $(\tau, \delta R_\tau b_\tau) = (0, 0)$, the household first order condition coincides with the planner's. Thus, any scheme $(\tau, w) > 0$ is *feasible* under the $(0, 0)$ fiscal and it is this policy that obtains the maximum utility for the household. Thus, the government will choose to set $\delta_\tau = 0$. □

corollary 1. *If $g_t = 0 \quad \forall t \geq \tau$ for some τ, then any sustainable plan must have $b_\tau = 0$.*

Claim 10.2. *If $b_t = 0 \quad \forall t = \tau, \tau + 1, ...$, then any sustainable plan must have default on $\tau - 1$ bonds.*

Proof. Suppose that pairing (σ, f) defines a sustainable plan for the economy in which i) $b_t = 0$ for all $t = \tau, \tau+1, ...$ and ii) there exists positive lending for all previous periods; that is, $b_t > 0 \quad \forall t = 0, 1, ..., \tau - 1$. The government faces the trigger mechanism which specifies that $b_{t+1} = 0$ for all future periods, given a deviation from policy in period t. Now, consider the government in period $\tau - 1$. It seeks to solve

$$\max_{\{\delta_t, \tau_t\}_{t=\tau-1}^{\infty}} \quad \sum_{t=\tau-1}^{\infty} \beta^{t-(\tau-1)} u(c_t, l_t) \tag{1}$$

$$s.t. \quad \delta_{\tau-1} R_{\tau-1} b_{\tau-1} + g_\tau \leq \tau_{\tau-1} l_{\tau-1}$$

$$s.t. \quad g_t \leq \tau_t l_t \qquad \forall t = \tau, \tau + 1, ...$$

If the government deviates in period $\tau - 1$, the trigger mechanism relegates the government to optimizing household welfare with respect to the government budget constraint

$$g_t \leq \tau_t l_t \qquad \forall t = \tau, \tau + 1, ...$$

which is no different than the original problem (1). Thus, the trigger mechanism does not constrain the government from optimizing in period $\tau - 1$. Given this, and through the same distortionary tax logic of the former proof, the government finds it optimal to maximize household welfare by setting $(\delta_{\tau-1}, \tau_{\tau-1}) = (0, 0)$.

Given this, the household allocation rule f would optimally set $b_{\tau-1} = 0$ in period $\tau - 2$. This contradicts the plan with $b_t > 0$ for $t = 0, 1, ..., \tau - 1$ being a sustainable one. Thus, a sustainable plan will have default in period $\tau - 1$. □

Through the same proof technique and using induction, we arrive at the following:

corollary 2. *If If $b_t = 0 \quad \forall t = \tau, \tau + 1, ...$, then any sustainable plan must also have default at period $t = 0$.*[42]

Thus, if the government faces a trigger mechanism of no borrowing autarky, and has a finite stream of expenditures (stopping at some date τ), then any sustainable plan must have date 0 default on any positive government debt.

[42] This follows through inductive reasoning.

F) Conjecture conditions on the stochastic process for government consumption, and on the discount fact so that the Ramsey outcome is sustainable. Try to prove your conjecture for extra credit.

In Chari and Kehoe (1990), the environment was static in the sense that there was not a state variable, such as bonds, that created a time-dependence. In that case, a Ramsey equilibrium would induce a constant utility payoff for all periods U^{RP} whereas the trigger strategy would consist of a one-period deviation utility $U^d(g)$ and a future, constant utility level V^d. Thus, for Ramsey plans to be sustainable, we must observe

$$\frac{1}{1-\beta}U^{RP} \geq U^d(g) + \frac{\beta}{1-\beta}V^d$$
$$\Rightarrow \frac{\beta}{1-\beta}[U^{RP} - V^d] \geq U^d(g) - U^{RP}.$$

Thus, as the discount factor gets sufficiently high, the Ramsey plan will be a sustainable plan. Further, if any plan is sustainable at discount β, it will be sustainable for discount factor $\beta' \in [\beta, 1]$.

Alternative Setup: Let the government consumption process be deterministic such that

$$g_t = \begin{cases} g_H, & \text{even periods} \\ 0, & \text{odd periods} \end{cases}$$

where the government can default on debt at any time and the government raises revenues by levying a proportional tax τ_t on labor income.

G) Define competitive equilibrium, set up the Ramsey problem and define a sustainable equilibrium.

A TDCE is a sequence of household allocation rules $Z^H = \{c_t, l_t, b_{t+1}\}_{t=0}^{\infty}$, a firm production plan $Z^F = \{l_t^f\}_{t=0}^{\infty}$, a government policy $\{g_t, \tau_t, \delta_t, b_{t+1}^g\}_{t=0}^{\infty}$ and prices $\{q_t, w_t\}_{t=0}^{\infty}$ such that the household, taking prices as given, solves

$$\max_{Z^H} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$
$$s.t. \quad c_t + b_{t+1} \leq (1-\tau_t) w_t l_t + (1-\delta_t) q_t b_t$$
$$s.t. \quad b_0 > 0, \quad b_{t+1} \in [-D, D], \quad c_t \geq 0 \ \ \forall t,$$

the firm, taking prices as given, solves

$$\max_{Z^F} l_t^f - w_t l_t^f \quad \forall t,$$

the government budget constraint holds:

$$g_t + (1-\delta_t) q_t b_t = \tau_t w_t l_t + b_{t+1} \quad \forall t$$

and markets clear:

$$c_t + g_t = l_t \quad \text{and} \quad b_t^g = b_t \quad \text{and} \quad l_t^f = l_t \quad \forall t.$$

In this model, a *Ramsey equilibrium* is a policy $\pi = (\pi_0, \pi_1, ...)$ and a household allocation rule $f = (f_1, f_2, ...)$ that satisfy

i. Policy π maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t(\pi), l_t(\pi))$ subject to GBC $\forall t$,

ii. $\forall \pi'$, the allocation rule $f(\pi')$ maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$, subject to HH budget constraint and GBC.

where i says the government policy maximizes date-0 welfare, and ii says that f is a best-response function to government policies.

Definition 10.4. *Let R_t denote the value of government surplus: $R_t = u_c(\tau_t l_t - g_t)$. Given the HH first order condition and resource feasibility. $R_t = u_c c_t + u_l l_t$.*

Proof.

$$\begin{aligned} R_t &= u_c(\tau_t l_t - l_t + c_t) \\ &= u_c c_t + u_c l_t(\tau_t - 1) \\ &= u_c c_t + u_l l_t \end{aligned}$$

□

Proposition 10.2. *In the Ramsey problem, consumption and labor allocations solve*

$$\sum_t \beta^t u(c_t, l_t)$$
$$s.t. c_t + g_t = l_t \tag{1}$$
$$s.t. \sum_t \beta^t R_t \geq min\{0, u_c(0)(1-\delta_0)b_0\} \tag{2}$$

The last condition is the implementability constraint. It requires that the value of government surpluses exceeds the value of initial debt. The *min* operator further requires that the government default on initial debt, if positive.

Proof. Subtract the consumer's budget constraint by the government's budget constraint to get the resource feasibility condition (1):

$$c_t + b_{t+1} - [\tau_t w_t l_t + b_{t+1}] = (1-\tau_t)w_t l_t + (1-\delta_t)q_t b_t - [g_t + (2-\delta_t)q_t b_t]$$
$$\Rightarrow c_t + g_t = w_t l_t = l_t \qquad \text{(imposing equilibrium wage)}$$

For condition (2), multiply the household budget constraint by $\beta^t u_c(t)$ and sum over time:

$$\sum_{t=0}^{\infty} \beta^t u_c(t)[c_t - (1-\tau_t)l_t + b_{t+1}] = \sum_{t=0}^{\infty} \beta^t u_c(t)(1-\delta_t)q_t b_t$$
$$\Rightarrow \sum_{t=0}^{\infty} \beta^t [u_c(t)c_t + u_l(t)l_t + u_c(t)b_{t+1}] = \sum_{t=0}^{\infty} \beta^t u_c(t)(1-\delta_t)q_t b_t \qquad (\text{by } -\tfrac{u_l(t)}{u_c(t)} = 1-\tau_t)$$
$$\Rightarrow \sum_{t=0}^{\infty} \beta^t [u_c(t) + u_l(t)] = u_c(0)(1-\delta_0)b_0 + \sum_{t=0}^{\infty} \beta^{t+1} u_c(t+1)(1-\delta_{t+1})q_{t+1}b_{t+1} - \beta^t u_c(t)b_{t+1} \quad (\text{assuming } q_0 = 1)$$
$$= u_c(0)(1-\delta_0)b_0 + \sum_{t=0}^{\infty} \beta^t b_{t+1}[\beta u_c(t1)(1-\delta_{t+1})q_{t+1} - u_c(t)]$$
$$= u_c(0)(1-\delta_0)b_0. \qquad (\text{by } u_c(t) = \beta(1-\delta_{t+1}q_{t+1}u_c(t+1)))$$

Further, if the government has positive initial debt, it is optimal to default (i.e. $\delta_0 = 0$) as to not distort household labor incentives. Thus,

$$u_c(0)(1-\delta_0)b_0 > 0 \underset{\sigma}{\implies} \delta_0 = 1$$

for a total default on initial debt. As a result, we have $\sum_{t=0}^{\infty} \beta^t [u_c(t)+u_l(t)] = \sum_{t=0}^{\infty} \beta^t R_t \geq min\{0, u_c(0)(1-\delta_0)b_0\}$, proving condition (2). □

Lastly, refer to **Definition 10.3** for a definition of sustainable equilibrium.

Disclaimer This section is more or less a regurgitation (an ugly word for an ugly answer) of Chari and Kehoe's older papers. It doesn't directly answer the question but was the best I had, going in to the prelims.

H) What is the worst sustainable equilibrium, without commitment? Show that the best sustainable equilibrium solves a programming problem. Show that the key constraint is any allocations must satisfy a sustainability constraint where the RHS of constraint is utility associated with best one shot deviation plus the sum of discounted utilities assocaited with the worst continuation equilibrium. Show that if g_H is sufficiently large, the Ramsey outcomes are sustainable in the best sustainable equilibrium.

In what follows, we will proceed in defining sequential equilibrium in an environment with trigger-type strategies, as done by Chari, Kehoe (1990) and Chari, Kehoe (1993). In this game, bonds act as a state variable which creates a time-dependence between periods; thus, this setting cannot be fully analyzed as a repeated game. For this, we will first define a Markov equilibrium and show that sustainable plans can be implemented through a *revert-to-Markov* equilibrium concept.

Now, we begin to describe the properties that construct a Markov equilibrium. Let the *Markov problem* be

$$\begin{aligned}
V_t(b_t) =& max \sum_{s=t}^{\infty} \beta^{s-t} u(c_s, l_s) \\
& s.t. -\frac{u_l(c_s, l_s)}{u_c(c_s, l_s)} = (1-\tau_s) \forall s = t, t+1, ... && \text{(FOC1)} \\
& s.t. u_c(c_s, l_s) = \beta(1-\delta_{t+1}) R_{s+1} u_c(c_{s+1}, l_{s+1}) \quad \forall s = t, t+1, ... && \text{(FOC2)} \\
& s.t. b_{t+1} \in [-D, D] && \text{(No Ponzi)} \\
& s.t. \sum_{s=r}^{\infty} \beta^{s-r} u(c_s, l_s) \geq V_r(b_r) \quad \forall r = t+1, t+2, ... && \text{(Consistency)} \\
& s.t. \text{HHBC}_s \text{ and } \text{GBC}_s \ \forall s = t, t+1, ... && \text{(Budget Constraint)}
\end{aligned}$$

where the Consistency condition requires that the plan chosen at date s will still be the solution to the date s value function when that date arrives. Note that at date 0, this gives the Ramsey allocations and policies. Consider the second programming problem: the *Household problem*:

$$\begin{aligned}
W_t(b_t, \pi_t) =& max \sum_{s=t}^{\infty} \beta^{s-t} u(c_s, l_s) \\
& s.t. FOC1, FOC2, \text{No Ponzi, Consistency, HHBC} \\
& s.t. \text{Government Budget Constraint}_s \ \forall s = t+1, t+2, ...
\end{aligned}$$

This is identical to the Markov problem, except for the fact that the date t government budget constraint does <u>not</u> need to be met. Thus, the problem allows for deviation in the current period. The Markov problem is used to construct a government policy σ^m and the Household problem is used to construct a consumer allocation f^m. This formulation can allow for more than one solution, as well.

Claim 10.3. *The policy plans (σ^m, f^m), constructed from solutions to the Markov problem and Household problem, are a Markov equilibrium. Further, these form a sustainable equilibrium.*

Proof. Both the Markov problem and the Household problem are recursive in nature. For the consumer, given some history $h_t = (h_{t-1}, \pi_t)$, maximize their utility subject to their resource constraints at all future dates, where future government policies evolve according to σ_m. The consumer chooses allocations with respect to its first order conditions FOC1, FOC2 and no ponzi conditions. Further, from date $t+1$ onward, the household allocation rules coincide with the Markov problem at date $t+1$, given inherited debt b_t, as a solution to the Household problem. Thus, there is no incentive to deviate. Now, consider the government at any history h_{t-1}. Note that

$$V_t(b_t) = \max_{\pi_t} W_t(b_t, \pi_t)$$

In this sense, there is an overlap in the incentives of both the government and household. Thus, there does not exist another one-shot deviation or policy that improves welfare of the household, given f^m. □

When, in a given period (or period 0), there is zero initial/entering debt, the Markov problem can be written as

$$V_0(b_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \tag{*}$$
$$s.t. c_t + g_t = l_t \quad \forall t$$
$$s.t. \sum_{t=0}^{\infty} \beta^t R_t \geq 0$$
$$s.t. \sum_{s=r}^{\infty} \beta^s R_s \leq 0 \quad \forall r = 1, 2, ...$$

requiring that the present value of all future government surpluses be nonpositive. It can be shown that there exists a debt sequence such that, a solution to $(*)$ coincides with the solution to the Markov problem. In the example, we will use these constraints to check that a plan satisfies the Markov problem.

Now, we consider a modified version of Markov equilibria: the *Revert-to-Markov equilibria.* This is an analogue to trigger strategies in repeated games. The mechanism works as follows: given some sequence (π, x), with policies $(\pi_0, \pi_1, ..., \pi_{t-1})$ up until date $t-1$, consider a deviation at time period t: $\hat{\pi}_t$. The allocation rule for date t is given by the solution to the Household problem $W_t(b_t, \hat{\pi}_t)$. Further, for any policy π_{t+1}, policies at date $t+1$ are solutions to $W_{t+1}(b^m_{t+1}, \pi_{t+1})$, and so forth in a recursive fashion. The reversion policy for the government works analogously.

Claim 10.4. *An arbitrary sequence (π, x) can be supported by a revert-to-Markov plan if and only if i) it is attainable under commitment (i.e. it maximizes date 0 utility subject to household and government budget constraints) and ii) for every t, the following inequality holds:*

$$\sum_{s=t}^{\infty} \beta^s u(c_s, l_s) \geq V_t(b_t).$$

Proof. Given attainable under commitment, the continuation of x is optimal given the continuation of π and vice versa. Further, for the government, the best one-shot deviation is the Markov strategy, and condition *ii)* shows that the government will choose to not deviate. Thus, a plan is sustainable under the trigger strategy if it maximizes date zero utility (subject to constraints) and offers weakly greater utility than the Markov problem at each date t. □

Claim 10.5. *In any revert-to-Markov equilibrium, the value of debt is nonpositive at each date. Further, when $b_0 = 0$, the allocation under the revert-to-Markov equilibrium are unique.*

First, consider the following scenario:

Odd Spending

For $g_t = 0$ for t even and $g_t = \gamma$ for t odd, under the Ramsey plan, the budget is balanced in a two-period cycle, such that

$$R(0) + \beta R(\gamma) = 0$$

Assume that the utility function is such that $R(g)$ is decreasing, which implies that $R(0)$ is positive and $R(\gamma)$ is negative. Thus, the government optimally smooths distortions by running a surplus in peacetime and deficit in wartime. For t even, we have $\sum_{r=t}^{\infty} \beta^r R(g) = 0$ and for t odd, we have $\sum_{r=t}^{\infty} \beta^r R(g_r) = \beta^t R(\gamma) < 0$. This implies the Ramsey allocations solve $(*)$ which implies that for a certain debt sequence, they are also sustainable. In even t, issue $b_{t+1} = \frac{R(\gamma)}{U_c(c(\gamma), l(\gamma))} < 0$. For t odd, issue $b_{t+1} = 0$. Essentially, the government runs a slight surplus in peacetime to make loans and pay for wartime spending through the return on those loans. Now, consider our example:

Even Spending

Given the claim above for a sustainable plan wit Revert-to-Markov strategies, the worst sustainable plan is one which is i) attainable under commitment and ii) offers weakly higher continuation utility, compared to the Markov problem at all future dates. Thus, if the utility associated with the Markov equilibrium is a lower bound, the Markov plan is the worst sustainable equilibrium.

In the problem, for $g_t = \gamma$ for t even and $g_t = 0$ for t odd, under the Ramsey plan, the budget is balanced in a two-period cycle, such that

$$R(\gamma) + \beta R(0) = 0$$

Since $R(g)$ is decreasing, this implies that $R(0)$ is positive and $R(\gamma)$ is negative. Thus, the government optimally smooths distortions by running a surplus in peacetime and deficit in wartime. For t even, we have $\sum_{r=t}^{\infty} \beta^r R(g_r) = \beta^t R(0) > 0$ and for t odd, we have $\sum_{r=t}^{\infty} \beta^r R(g_r) = 0$. In even t, issue $b_{t+1} = \frac{R(0)}{U_c(c(0), l(0))} > 0$. For t odd, issue $b_{t+1} = 0$. This implies a peacetime surplus and wartime deficit, but at date 0, it issues positive debt. The government will find it optimal to default on this debt at date 1 and follow the Markov plan: this will give the Ramsey allocations of the corresponding problem above. Thus, positive debt is not sustainable at date 0 (by the revert to Markov plan) and the government must finance the initial wartime spending through distortionary taxes. This is the best sustainable equilibrium: it features a balanced government budget in the initial period, and then the implementation of the Ramsey problem, starting in period $t = 1$, which is the peacetime period.

condition *ii*) of the claim above, says that for any date t, the continuation utility of a plan must exceed the continuation utility of the Markov problem. If it does not, then agents have the incentive to deviate in period t and then play within the confines of the Markov equilibrium. Thus, given a sustainable plans induced stream $(\bar{c}, \bar{l})_{t=0}^{\infty}$, it must satisfy the date t constraint

$$\sum_{s=t}^{\infty} \beta^{s-r} u(\overline{c_s}, \overline{l_s}) \geq \max_{\pi'} \; u(c_t^*, l_t^*) + \beta[\sum_{s=t+1}^{\infty} \beta^{2(s-(t+1))} V_{M1} + \beta^{2[s-(t+1)]+1} V_{M2}]$$

for all t, where (c^*, l^*) represents the quantities associated with the best one-shot deviation in period t and V_{M1} and V_{M2} represent the period utilities associated with the Markov problem in peacetime and wartime spending, depending upon the date t of the deviation.

For implementing the Ramsey equilibrium as the best sustainable equilibrium, we will require a different deviation policy (other than the Revert-to-Markov strategy). If there exists a multiplicity of sustainable equilibrium, we can define trigger strategies that revert to this level of present value welfare. For sufficiently high government spending and sufficiently high discount factors, the government would find it optimal to maintain positive debt with zero default. It would do this because the trigger strategy would outweigh the benefit of such a one-shot deviation.

11 Appendix

11.1 Transversality Condition

Often overlooked and misunderstood is the transversality condition that we impose on maximization problems when the model is one of infinite time. This condition can be interpreted as a boundary condition of sorts, and is used as a sufficient (and generally necessary) condition to ensure a solution to the maximization program under consideration. As some motivation for its usage, consider the following problem

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(F(k_t) - k_{t+1})$$

subject to the constraint $k_{t+1} \in [0, F(k_t)]$, where initial capital k_0 is given and $T < \infty$. Given a strictly increasing utility function, the agent solving this problem wishes to consume as much as possible. Thus, in period T, the agent optimally selects $k_{T+1} = 0$ and $c_T = F(k_T)$. This gives us a boundary condition for the choice of capital in the last period. In this model, we can derive the Euler equations

$$u'(F(k_t) - k_{t+1}) = \beta u'(F(k_{t+1}) - k_{t+2})F'(k_{t+1}) \quad \forall t = 0, 1, ..., T-1$$

This equation applies to all time periods; thus, we *would* have a system of t equations in $T+1$ unknowns, but the end condition $K_T + 1$ reduces that to an identified system, which can yield a unique solution. So, what becomes of us when we move on to the problem

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(F(k_t) - k_{t+1})$$

with the same constraint set? We have the same Euler equation conditions, but now we have an infinite series of choices and no clear *end of time* boundary condition for capital. The solution is to introduce the transversality condition (TVC) which acts as the infinite time analogue to the previous boundary condition. What does K_{T+1} mean for the finite time model? It more or less implies that value of capital in time $T+1$ is zero for the agent, in terms of its present value marginal benefit. Because of this, the agent optimally sets it to zero. In mathematical terms, it basically means

$$\beta^{T+1} u'(F(k_{T+1}) - k_{T+2})F'(K_{T+1}) \cdot k_{T+1} = 0.$$

Let's unpack this. β^{T+1} puts this period $T+1$ return in terms of present value. $u'(F(k_{T+1}) - k_{T+2})F'(K_{T+1})$ provides the marginal utility of increasing capital investment into period $T+1$, where its magnitude is measured by the marginal increase in output (and hence marginal increase in $T+1$ consumption). Multiplied by total k_{T+1} gives the value of the k_{T+1} capital stock, measured in time 0 utility. Thus, in infinite time, we impose the transversality condition

$$\mathit{lim}_{t\to\infty} \beta^t u'(F(k_t) - k_{t+1})F'(k_t) \cdot k_t = 0. \tag{TVC}$$

Given certain assumptions on the utility function and economic environment (Theorem 4.15 of SLP), the Euler equations <u>and</u> TVC are necessary and sufficient conditions for a solution to the sequence problem. What does this mean? There may be a variety of initial capital positions k_0 and capital sequences that satisfy the Euler equations. These will not be optimal unless they satisfy the TVC, as well. This is a mathematical result but should also be an intuitive one, as well: it makes sense that the marginal value of capital investment is insignificant to the agent, when measured in terms of date 0 utility. In some circumstances, the TVC is represented by the condition

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0$$

where λ_t is the multiplier on the feasibility constraint from the planner problem. This is equivalent to the above stated TVC:

$$\begin{aligned}
\lim_{t\to\infty} \lambda_t k_{t+1} &= \lim_{t\to\infty} \beta^t u'(F(k_t) - k_{t+1})k_{t+1} \\
&= \lim_{t\to\infty} \beta^{t-1} u'(F(k_{t-1}) - k_t)k_t \\
&= \lim_{t\to\infty} \beta^{t-1}[\beta u'(F(k_t) - k_{t+1})F'(k_t)]k_t \qquad \text{(Euler Equation Sub)} \\
&= \lim_{t\to\infty} \beta^t u'(F(k_t) - k_{t+1})F'(k_t) \cdot k_t = 0.
\end{aligned}$$

Transversality conditions can also be imposed upon other agent decisions within a model when there are more choice variables with inter-temporal ramifications.

References

[1] Chari, V.V. and Kehoe, Patrick J. (1990) *Sustainable Plans. Journal of Political Economy*, vol. 98 (4), 783-802.

[2] Huggett, Mark. (1993) *The Risk-Free Rate in Heterogenuous-Agent, Incomplete-Insurance Economies. Journal of Economic Dynamics and Control*, vol. 17, 953-969.

[3] Ljungqvist, Lars and Sargent, Thomas J. *Recrusive Macroeconomic Theory (3rd Edition)*. Cambridge, Massachusetts: MIT Press, 2012.

[4] Lucas Jr., Robert E. and Stokey, Nancy. (1986) *Money and Interest In a Cash-in-Advance Economy. Discussion Paper N0. 628.*

[5] Stokey, Nancy and Lucas Jr., Robert E. *Recursive Methods in Economic Dynamics*. Cambridge, Massachusetts: Harvard University Press, 1989.

[6] Woodford, Michael. (1994) *Monetary Policy and Price Level Determinacy in a Cash-in-Advance Economy. Journal of Economic Theory*, vol. 4, 345-380.